THE CORONA THEOREM FOR THE DRURY-ARVESON HARDY SPACE AND OTHER HOLOMORPHIC BESOV-SOBOLEV SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

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ABSTRACT. We prove that the multiplier algebra of the Drury-Arveson Hardy space H_n^2 on the unit ball in $\mathbb{C}^n$ has no corona in its maximal ideal space, thus generalizing the Corona Theorem of L. Carleson to higher dimensions. This result is obtained as a corollary of the Toeplitz corona theorem and a new Banach space result: the Besov-Sobolev space B_p^σ has the "baby corona property" for all $\sigma \geq 0$ and $1 < p < \infty$. In addition we obtain infinite generator and semi-infinite matrix versions of these theorems.

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1. Introduction

In 1962 Lennart Carleson demonstrated in [12] the absence of a corona in the maximal ideal space of $H^{\infty}(\mathbb{D})$ by showing that if $\{g_j\}_{j=1}^{N}$ is a finite set of functions in $H^{\infty}(\mathbb{D})$ satisfying

$$\sum_{j=1}^{N} |g_j(z)| \geq c > 0, \qquad z \in \mathbb{D}, \tag{1.1}$$

then there are functions $\{f_j\}_{j=1}^{N}$ in $H^{\infty}(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z)\, g_j(z) = 1, \qquad z \in \mathbb{D}, \tag{1.2}$$

In 1968 Fuhrmann [14] extended Carleson's corona theorem to the finite matrix case. In 1980 Rosenblum [23] and Tolokonnikov [27] proved the corona theorem for infinitely many generators $N = \infty$. This was further generalized to the one-sided infinite matrix setting by Vasyunin in 1981 (see [28]). Finally Treil [30] showed in 1988 that the generalizations stop there by producing a counterexample to the two-sided infinite matrix case.

Hörmander noted a connection between the corona problem and the Koszul complex, and in the late 1970's Tom Wolff gave a simplified proof using the theory of the $\overline{\partial}$ equation and Green's theorem (see [15]). This proof has since served as a model for proving corona type theorems for other Banach algebras.

While there is a large literature on corona theorems in one complex dimension (see e.g. [19]), progress in higher dimensions has been limited. Indeed, apart from the simple cases in which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no corona theorem has been proved until now in higher dimensions. Instead, partial results have been obtained, such as the beautiful Toeplitz corona theorem for Hilbert function spaces with a complete Nevanlinna-Pick kernel, the H^p corona theorem on the ball and polydisk, and results restricting N to 2 generators in (1.1) (the case $N = 1$ is trivial). In particular, Varopoulos [35] published a lengthy classic paper in an unsuccessful attempt to prove the corona theorem for the multiplier algebra $H^{\infty}(\mathbb{B}_n)$ of the classical Hardy space $H^2(\mathbb{B}_n)$ of holomorphic functions on the ball with square integrable boundary values. His BMO estimates for solutions with $N = 2$ generators remain unimproved to this day. We will discuss these partial results in more detail below.

Our main result is that the corona theorem, namely the absence of a corona in the maximal ideal space, holds for the multiplier algebra $M_{H_n^2}$ of the Hilbert space H_n^2, the celebrated Drury-Arveson Hardy space on the ball in n dimensions.

Theorem 1. *If $\{g_j\}_{j=1}^{N}$ is a finite set of functions in $M_{H_n^2}$ satisfying (1.1), then there are functions $\{f_j\}_{j=1}^{N}$ in $M_{H_n^2}$ satisfying (1.2).*

In many ways H_n^2, and not the more familiar space $H^2(\mathbb{B}_n)$, is the natural generalization to higher dimensions of the classical Hardy space on the disk. For example, H_n^2 is universal among Hilbert function spaces with the complete Pick property, and its multiplier algebra $M_{H_n^2}$ is the correct home for the multivariate von Neumann inequality (see e.g. [9]). See Arveson [8] for more on the space H_n^2, including the model theory of n-contractions.

More generally, the corona theorem holds for the multiplier algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ of the Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, $0 \leq \sigma \leq \frac{1}{2}$, on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. The space $B_2^\sigma(\mathbb{B}_n)$ consists roughly of those holomorphic functions f whose derivatives of order $\frac{n}{2} - \sigma$ lie in the classical Hardy space $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{B}_n)$, and is normed by

$$\|f\|_{B_2^\sigma(\mathbb{B}_n)} = \left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^2 + \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} R^m f(z) \right|^2 d\lambda_n(z) \right\}^{\frac{1}{2}},$$

for some $m > \frac{n}{2} - \sigma$ where $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$ is the radial derivative. In particular $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$. Finally, we also obtain semi-infinite matrix versions of these results.

Note: Our techniques also yield BMO estimates for the $H^\infty(\mathbb{B}_n)$ corona problem, which will appear elsewhere.

2. The corona problem in $\mathbb{C}^n$

Let X be a Hilbert space of holomorphic functions in an open set Ω in $\mathbb{C}^n$ that is a reproducing kernel Hilbert space with a *complete irreducible Nevanlinna-Pick kernel* (see [9] for the definition). The following *Toeplitz corona theorem* is due to Ball, Trent and Vinnikov [10] (see also Ambrozie and Timotin [2] and Theorem 8.57 in [9]).

For $f = (f_\alpha)_{\alpha=1}^N \in \oplus^N X$ and $h \in X$, define $\mathbb{M}_f h = (f_\alpha h)_{\alpha=1}^N$ and

$$\|f\|_{Mult(X, \oplus^N X)} = \|\mathbb{M}_f\|_{X \to \oplus^N X} = \sup_{\|h\|_X \leq 1} \|\mathbb{M}_f h\|_{\oplus^N X}.$$

Note that $\max_{1 \leq \alpha \leq N} \|\mathcal{M}_{f_\alpha}\|_{M_X} \leq \|f\|_{Mult(X, \oplus^N X)} \leq \sqrt{\sum_{\alpha=1}^N \|\mathcal{M}_{f_\alpha}\|_{M_X}^2}$.

Toeplitz corona theorem: Let X be a Hilbert function space in an open set Ω in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\delta > 0$ and $N \in \mathbb{N}$. Then $g_1, ..., g_N \in M_X$ satisfy the following "baby corona property"; for every $h \in X$, there are $f_1, ..., f_N \in X$ such that

$$\begin{aligned} \|f_1\|_X^2 + ... + \|f_N\|_X^2 &\leq \frac{1}{\delta} \|h\|_X^2, \\ g_1(z) f_1(z) + ... + g_N(z) f_N(z) &= h(z), \qquad z \in \Omega, \end{aligned} \tag{2.1}$$

if and only if $g_1, ..., g_N \in M_X$ satisfy the following "multiplier corona property"; there are $\varphi_1, ..., \varphi_N \in M_X$ such that

$$\begin{aligned} \|\varphi\|_{Mult(X, \oplus^N X)} &\leq 1, \\ g_1(z) \varphi_1(z) + ... + g_N(z) \varphi_N(z) &= \sqrt{\delta}, \qquad z \in \Omega. \end{aligned} \tag{2.2}$$

The *baby corona theorem* is said to hold for X if whenever $g_1, ..., g_N \in M_X$ satisfy

$$|g_1(z)|^2 + ... + |g_N(z)|^2 \geq c > 0, \qquad z \in \Omega, \tag{2.3}$$

then $g_1, ..., g_N$ satisfy the baby corona property (2.1). The Toeplitz corona theorem thus provides a useful tool for reducing the multiplier corona property (2.2) to the more tractable, but still very difficult, baby corona property (2.1) for multiplier algebras $M_{B_p^\sigma(\mathbb{B}_n)}$ of certain of the Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ when $p = 2$ - see below. The case of $M_{B_p^\sigma(\mathbb{B}_n)}$ when $p \neq 2$ must be handled by more classical methods and remains largely unsolved.

Remark 1. *A standard abstract argument applies to show that the absence of a corona for the multiplier algebra M_X, i.e. the density of the linear span of point evaluations in the maximal ideal space of M_X, is equivalent to the following assertion: for each finite set $\{g_j\}_{j=1}^N \subset M_X$ such that (2.3) holds for some $c > 0$, there are $\{\varphi_j\}_{j=1}^N \subset M_X$ and $\delta > 0$ such that condition (2.2) holds. See for example Lemma 9.2.6 in [19] or the proof of Criterion 3.5 on page 39 of [25].*

2.1. **The Baby Corona Theorem.** To treat $N > 2$ generators in (2.1), it is just as easy to treat the case $N = \infty$, and this has the advantage of not requiring bookkeeping of constants depending on N. We will

(1) use the Koszul complex for infinitely many generators, and
(2) invert higher order forms in the $\overline{\partial}$ equation, and
(3) devise new estimates for the Charpentier solution operators for these equations including,
    (a) the use of sharp estimates on Euclidean expressions $\left|\left(\overline{w-z}\right)\frac{\partial}{\partial \overline{w}} f\right|$ in terms of the invariant derivative $\left|\widetilde{\nabla} f\right|$ (see Proposition 4),
    (b) the use of the exterior calculus together with the explicit form of Charpentier's solution kernels in Theorems 4 and 6 to handle *rogue* Euclidean factors $\overline{w_j - z_j}$ (see Section 8), and
    (c) the application of generalized operator estimates of Schur type in Lemma 10 to obtain appropriate boundedness of solution operators.

In addition to these novel elements in the proof, we make crucial use of the beautiful integration by parts formula of Ortega and Fabrega [20], and in order to obtain ℓ^2-valued results, we use the clever factorization of the Koszul complex in Andersson and Carlsson [4] but adapted to ℓ^2.

Notation 1. *For sequences $f(z) = (f_i(z))_{i=1}^{\infty} \in \ell^2$ we will write*

$$|f(z)| = \sqrt{\sum_{i=1}^{\infty} |f_i(z)|^2}.$$

When considering sequences of vectors such as $\nabla^m f(z) = (\nabla^m f_i(z))_{i=1}^{\infty}$, the same notation $|\nabla^m f(z)| = \sqrt{\sum_{i=1}^{\infty} |\nabla^m f_i(z)|^2}$ will be used with $|\nabla^m f_i(z)|$ denoting the Euclidean length of the vector $\nabla^m f_i(z)$. Thus the symbol $|\cdot|$ is used in at least three different ways; to denote the absolute value of a complex number, the length of a finite vector in $\mathbb{C}^N$ and the norm of a sequence in ℓ^2. Later it will also be used to denote the Hilbert-Schmidt norm of a tensor, namely the square root of the sum of the squares of the coefficients in the standard basis. In all cases the meaning should be clear from the context.

Recall that $B_p^{\sigma}(\mathbb{B}_n; \ell^2)$ consists of all $f = (f_i)_{i=1}^{\infty} \in H(\mathbb{B}_n; \ell^2)$ such that

(2.4)
$$\|f\|_{B_p^{\sigma}(\mathbb{B}_n;\ell^2)} \equiv \sum_{k=0}^{m-1} \left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} < \infty,$$

for some $m > \frac{n}{p} - \sigma$. By Proposition 1 below (see also [11]), the right side is finite for some $m > \frac{n}{p} - \sigma$ if and only if it is finite for all $m > \frac{n}{p} - \sigma$. As usual we will write $B_p^{\sigma}(\mathbb{B}_n)$ for the scalar-valued space.

We now state our baby corona theorem for the ℓ^2-valued Banach spaces $B_p^\sigma(\mathbb{B}_n;\ell^2)$, $\sigma \geq 0$, $1<p<\infty$. Observe that for $\sigma<0$, $M_{B_p^\sigma(\mathbb{B}_n)} = B_p^\sigma(\mathbb{B}_n)$ is a subalgebra of $C\left(\overline{\mathbb{B}_n}\right)$ and so has no corona. The $N=2$ generator case of Theorem 2 when $\sigma \in \left[0,\frac{1}{p}\right) \cup \left(\frac{n}{p},\infty\right)$ and $1<p<\infty$ is due to Ortega and Fabrega [20], who also obtain the $N=2$ generator case when $\sigma=\frac{n}{p}$ and $1<p\leq 2$. See Theorem A in [20]. In [21] Ortega and Fabrega prove analogous results with scalar-valued Hardy-Sobolev spaces in place of the Besov-Sobolev spaces.

Let $\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ denote the norm of the multiplication operator $\mathbb{M}_g$ from $B_p^\sigma(\mathbb{B}_n)$ to the ℓ^2-valued Besov-Sobolev space $B_p^\sigma(\mathbb{B}_n;\ell^2)$.

Theorem 2. *Let $\delta>0$, $\sigma\geq 0$ and $1<p<\infty$. Then there is a constant $C_{n,\sigma,p,\delta}$ such that given $g=(g_i)_{i=1}^\infty \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ satisfying*

$$\begin{aligned} \|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} &\leq 1, \\ \sum_{j=1}^\infty |g_j(z)|^2 &\geq \delta^2>0, \qquad z\in\mathbb{B}_n, \end{aligned}$$

there is for each $h\in B_p^\sigma(\mathbb{B}_n)$ a vector-valued function $f\in B_p^\sigma(\mathbb{B}_n;\ell^2)$ satisfying

$$\begin{aligned} \|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} &\leq C_{n,\sigma,p,\delta}\|h\|_{B_p^\sigma(\mathbb{B}_n)}, \\ \sum_{j=1}^\infty g_j(z)f_j(z) &= h(z), \qquad z\in\mathbb{B}_n. \end{aligned} \tag{2.5}$$

Corollary 1. *Let $0\leq\sigma\leq\frac{1}{2}$. Then the Banach algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ has no corona, i.e. (2.1) implies (2.2). In particular this includes Theorem 1 that the multiplier algebra of the Drury-Arveson space $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$ has no corona (the one-dimensional case is Carleson's corona theorem), and also includes that the multiplier algebra of the n-dimensional Dirichlet space $\mathcal{D}(\mathbb{B}_n)=B_2^0(\mathbb{B}_n)$ has no corona (the one-dimensional case here is due to Tolokonnikov [29]).*

The corollary follows immediately from the finite generator case $p=2$ of Theorem 2 and the Toeplitz corona theorem (and Remark 1) since the spaces $B_2^\sigma(\mathbb{B}_n)$ have an irreducible complete Nevanlinna-Pick kernel when $0\leq\sigma\leq\frac{1}{2}$ ([7]).

We also have a semi-infinite matricial corona theorem.

Corollary 2. *Let $0\leq\sigma\leq\frac{1}{2}$. Let $\mathcal{H}_1$ be a finite m-dimensional Hilbert space and let $\mathcal{H}_2$ be an infinite dimensional separable Hilbert space. Suppose that $F\in\mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_1\to\mathcal{H}_2)}$ satisfies $\delta^2 I_m\leq F^*(z)F(z)\leq I_m$. Then there is $G\in\mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2\to\mathcal{H}_1)}$ such that*

$$\begin{aligned} G(z)F(z) &= I_m, \\ \|G\|_{\mathcal{M}_{B_2^\sigma(\mathbb{B}_n)(\mathcal{H}_2\to\mathcal{H}_1)}} &\leq C_{\sigma,n,\delta,m}. \end{aligned}$$

This corollary follows immediately from the case $p=2$ of Theorem 2 and the Toeplitz corona theorem together with Theorem (MCT) in Trent and Zhang [34]. See [34] for the notation used here. We already commented above on the special case of this corollary for the Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_1)=H^2(\mathbb{D})$ on the disk. The case $m=1$ of this corollary for the classical Dirichlet space $B_2^0(\mathbb{B}_1)=\mathcal{D}(\mathbb{D})$ on the disk is due to Trent [33]. It would be of interest to determine the dependence of the constants on p and δ in Theorem 2.

2.1.1. *Prior results.* In [4] Andersson and Carlsson solve the baby corona problem for $H^2(\mathbb{B}_n)$ and obtain the analogous (baby) H^p corona theorem on the ball $\mathbb{B}_n$ for $1 < p < \infty$ and with constants independent of the number of generators (see also Amar [1], Andersson and Carlsson [5],[3] and Krantz and Li [16]). Partial results on the corona problem restricted to $N = 2$ generators and BMO in place of L^∞ estimates have been obtained for $H^\infty(\mathbb{B}_n)$ (the multiplier algebra of $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{B}_n)$) by N. Varopoulos [35] in 1977. This classical corona problem remains open (Problem 19.3.7 in [24]), along with the corona problems for the multiplier algebras of $B_2^\sigma(\mathbb{B}_n)$, $\frac{1}{2} < \sigma < \frac{n}{2}$.

More recently in 2000 J. M. Ortega and J. Fabrega [20] obtain partial results with $N = 2$ generators in (2.1) for the algebras $M_{B_2^\sigma(\mathbb{B}_n)}$ when $0 \le \sigma < \frac{1}{2}$, i.e. from the Dirichlet space $B_2^0(\mathbb{B}_n)$ up to but not including the Drury-Arveson Hardy space $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$. To handle $N = 2$ generators they exploit the fact that a 2×2 antisymmetric matrix consists of just one entry up to sign, so that as a consequence the form Ω_1^2 in the Koszul complex below is $\overline{\partial}$-closed. The paper [20] by Ortega and Fabrega has proved to be of enormous influence in our work, as the basic groundwork and approach we use are set out there.

In [31] S. Treil and the third author obtain the H^p corona theorem on the polydisk $\mathbb{D}^n$ (see also Lin [18] and Trent [32]). The Hardy space $H^2(\mathbb{D}^n)$ on the polydisk fails to have the complete Nevanlinna-Pick property, and consequently the Toeplitz corona theorem only holds in a more complicated sense that a family of kernels must be checked for positivity instead of just one. As a result the corona theorem for the algebra $H^\infty(\mathbb{D}^n)$ on the polydisk remains open for $n \ge 2$. Finally, even the baby corona problems, apart from that for H^p, remain open on the polydisk.

2.2. **Plan of the paper.** We will prove Theorem 2 using the Koszul complex and a factorization of Andersson and Carlsson, an explicit calculation of Charpentier's solution operators, and generalizations of the integration by parts formulas of Ortega and Fabrega, together with new estimates for boundedness of operators on certain real-variable analogues of the holomorphic Besov-Sobolev spaces. Here is a brief plan of the proof.

We are given an infinite vector of multipliers $g = (g_i)_{i=1}^\infty \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ that satisfy $\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \le 1$ and $\inf_{\mathbb{B}_n}|g| \ge \delta > 0$, and an element $h \in B_p^\sigma(\mathbb{B}_n)$. We wish to find $f = (f_i)_{i=1}^\infty \in B_p^\sigma\left(\mathbb{B}_n;\ell^2\right)$ such that

(1) $\mathcal{M}_g f = g \cdot f = h$,
(2) $\overline{\partial} f = 0$,
(3) $\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \le C_{n,\sigma,p,\delta}\|h\|_{B_p^\sigma(\mathbb{B}_n)}$.

An obvious first attempt at a solution is

$$f = \frac{\overline{g}}{|g|^2}h,$$

since f obviously satisfies (1), can be shown to satisfy (3), but fails to satisfy (2) in general.

To rectify this we use the Koszul complex in Section 5, which employs *any* solution to the $\overline{\partial}$ problem on forms of bidegree $(0,q)$, $1 \le q \le n$, to produce a

correction term $\Lambda_g \Gamma_0^2$ so that

$$f = \frac{\overline{g}}{|g|^2} h - \Lambda_g \Gamma_0^2$$

now satisfies (1) and (2), but (3) is now in doubt without specifying the exact nature of the correction term $\Lambda_g \Gamma_0^2$.

In Section 3 we explicitly calculate Charpentier's solution operators to the $\overline{\partial}$ equation for use in solving the $\overline{\partial}$ problems arising in the Koszul complex. These solution operators are remarkably simple in form and moreover are superbly adapted for obtaining estimates in real-variable analogues of the Besov-Sobolev spaces in the ball. In particular, the kernels $K(w,z)$ of these solution operators involve expressions like

$$\frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q \overline{(w-z)}}{\triangle(w,z)^n}, \tag{2.6}$$

where

$$\sqrt{\triangle(w,z)} = \left| P_z(w-z) + \sqrt{1-|z|^2} Q_z(w-z) \right|$$

is the length of the vector $w-z$ shortened by multiplying by $\sqrt{1-|z|^2}$ its projection $Q_z(w-z)$ onto the orthogonal complement of the complex line through z. Also useful is the identity $\sqrt{\triangle(w,z)} = |1-w\overline{z}|\,|\varphi_z(w)|$ where φ_z is the involutive automorphism of the ball that interchanges z and 0; in particular this shows that $d(w,z) = \sqrt{\triangle(w,z)}$ is a quasimetric on the ball.

In Section 6.1 we introduce real-variable analogues $\Lambda^\sigma_{p,m}(\mathbb{B}_n)$ of the Besov-Sobolev spaces $B^\sigma_p(\mathbb{B}_n)$ along with ℓ^2-valued variants, that are based on the geometry inherent in the complex structure of the ball and reflected in the solution kernels in (2.6). In particular these norms involve modifications D of the invariant derivative $\widetilde{\nabla}$ in the ball:

$$Df(w) = \left(1-|w|^2\right) P_w \nabla + \sqrt{1-|w|^2} Q_w \nabla.$$

Three crucial inequalities are then developed to facilitate the boundedness of the Charpentier solution operators, most notably

$$\left| \overline{(z-w)}^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w) \right| \leq C \triangle(w,z)^{\frac{m}{2}} \left| \left(1-|w|^2\right)^{-m} \overline{D}^m F(w) \right|, \tag{2.7}$$

for $F \in H^\infty\left(\mathbb{B}_n;\ell^2\right)$, which controls the product of Euclidean lengths with Euclidean derivatives on the left, in terms of the product of the smaller length $\sqrt{\triangle(w,z)}$ and the larger derivative $\left(1-|w|^2\right)^{-1}\overline{D}$ on the right. We caution the reader that our definition of $\overline{D}^m$ is *not* simply the composition of m copies of $\overline{D}$ - see Definition 6 below.

In Section 4 we recall the clever integration by parts formulas of Ortega and Fabrega involving the left side of (2.7), and extend them to the Charpentier solution operators for higher degree forms. If we differentiate (2.6), the power of $\triangle(w,z)$ in the denominator can increase and the integration by parts in Lemma 3 below will temper this singularity on the diagonal. On the other hand the radial integration by parts in Corollary 3 below will temper singularities on the boundary of the ball.

In Section 7 we use Schur's Test to establish the boundedness of positive operators with kernels of the form

$$\frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^b \sqrt{\triangle(w,z)}^c}{|1-\overline{w}z|^{a+b+c+n+1}}.$$

The case $c=0$ is standard (see e.g. [36]) and the extension to the general case follows from an automorphic change of variables. These results are surprisingly effective in dealing with the ameliorated solution operators of Charpentier.

Finally in Section 8 we put these pieces together to prove Theorem 2.

The appendix collects technical proofs of formulas and modifications of existing proofs in the literature that would otherwise interrupt the main flow of the paper.

3. Charpentier's solution kernels for $(0,q)$-forms on the ball

In Theorem I.1 on page 127 of [13], Charpentier proves the following formula for $(0,q)$-forms:

Theorem 3. *For $q \geq 0$ and all forms $f(\xi) \in C^1\left(\overline{\mathbb{B}_n}\right)$ of degree $(0,q+1)$, we have for $z \in \mathbb{B}_n$:*

$$f(z) = C_q \int_{\mathbb{B}_n} \overline{\partial} f(\xi) \wedge \mathcal{C}_n^{0,q+1}(\xi,z) + c_q \overline{\partial}_z \left\{ \int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi,z) \right\}.$$

Here $\mathcal{C}_n^{0,q}(\xi,z)$ is a $(n,n-q-1)$-form in ξ on the ball and a $(0,q)$-form in z on the ball that is defined in Definition 2 below. Using Theorem 3, we can solve $\overline{\partial}_z u = f$ for a $\overline{\partial}$-closed $(0,q+1)$-form f as follows. Set

$$u(z) \equiv c_q \int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi,z)$$

Taking $\overline{\partial}_z$ of this we see from Theorem 3 and $\overline{\partial} f = 0$ that

$$\overline{\partial}_z u = c_q \overline{\partial}_z \left(\int_{\mathbb{B}_n} f(\xi) \wedge \mathcal{C}_n^{0,q}(\xi,z) \right) = f(z).$$

It is essential for our proof to explicitly compute the kernels $\mathcal{C}_n^{0,q}$ when $0 \leq q \leq n-1$. The case $q=0$ is given in [13] and we briefly recall the setup. Denote by $\triangle : \mathbb{C}^n \times \mathbb{C}^n \to [0,\infty)$ the map

$$\triangle(w,z) \equiv |1-w\overline{z}|^2 - \left(1-|w|^2\right)\left(1-|z|^2\right).$$

We compute that

$$\begin{aligned}
\triangle(w,z) &= 1 - 2\operatorname{Re} w\overline{z} + |w\overline{z}|^2 - \left\{1-|w|^2-|z|^2+|w|^2|z|^2\right\} \\
&= |w-z|^2 + |w\overline{z}|^2 - |w|^2|z|^2 \\
&= \left(1-|z|^2\right)|w-z|^2 + |z|^2\left(|w-z|^2-|w|^2\right) + |w\overline{z}|^2 \\
&= \left(1-|z|^2\right)|w-z|^2 + |z|^4 - 2\operatorname{Re}|z|^2 w\overline{z} + |w\overline{z}|^2 \\
&= \left(1-|z|^2\right)|w-z|^2 + |\overline{z}(w-z)|^2,
\end{aligned} \tag{3.1}$$

and by symmetry

$$\triangle(w,z) = \left(1-|w|^2\right)|w-z|^2 + |\overline{w}(w-z)|^2.$$

We also have the standard identity

$$\triangle(w,z) = |1-z\overline{w}|^2 |\varphi_w(z)|^2, \tag{3.2}$$

where

$$\varphi_w(z) = \frac{P_w(w-z) + \sqrt{1-|w|^2}\, Q_w(w-z)}{1-\overline{w}z}.$$

Thus we also have

$$\begin{aligned} \triangle(w,z) &= \left| P_w(z-w) + \sqrt{1-|w|^2}\, Q_w(z-w) \right|^2 \\ &= \left| P_z(z-w) + \sqrt{1-|z|^2}\, Q_z(z-w) \right|^2. \end{aligned} \tag{3.3}$$

It is convenient to combine the many faces of $\triangle(w,z)$ in (3.1), (3.2) and (3.3) in:

$$\begin{aligned} \triangle(w,z) &= |1-w\overline{z}|^2 - \left(1-|w|^2\right)\left(1-|z|^2\right) \\ &= \left(1-|z|^2\right)|w-z|^2 + |\overline{z}(w-z)|^2 \\ &= \left(1-|w|^2\right)|w-z|^2 + |\overline{w}(w-z)|^2 \\ &= |1-w\overline{z}|^2 |\varphi_w(z)|^2 \\ &= |1-w\overline{z}|^2 |\varphi_z(w)|^2 \\ &= \left| P_w(z-w) + \sqrt{1-|w|^2}\, Q_w(z-w) \right|^2 \\ &= \left| P_z(z-w) + \sqrt{1-|z|^2}\, Q_z(z-w) \right|^2. \end{aligned} \tag{3.4}$$

To compute the kernels $\mathcal{C}_n^{0,q}$ we start with the Cauchy-Leray form

$$\mu(\xi,w,z) \equiv \frac{1}{(\xi(w-z))^n} \sum_{i=1}^n (-1)^{i-1} \xi_i \left[\wedge_{j\neq i} d\xi_j\right] \wedge_{i=1}^n d(w_i - z_i),$$

which is a closed form on $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$ since with $\zeta = w-z$, μ is a pullback of the form

$$\nu(\xi,\zeta) \equiv \frac{1}{(\xi\zeta)^n} \sum_{i=1}^n (-1)^{i-1} \xi_i \left[\wedge_{j\neq i} d\xi_j\right] \wedge_{i=1}^n d\zeta_i,$$

which is easily computed to be closed (see e.g. 16.4.5 in [24]).

One then lifts the form μ via a section s to give a closed form on $\mathbb{C}^n \times \mathbb{C}^n$. Namely, for $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ one defines,

$$s^*\mu(w,z) \equiv \frac{1}{(s(w,z)(w-z))^n} \sum_{i=1}^n (-1)^{i-1} s_i(w,z) \left[\wedge_{j\neq i} ds_j\right] \wedge_{i=1}^n d(w_i - z_i).$$

Now we fix s to be the following section used by Charpentier:

$$s(w,z) \equiv \overline{w}(1-w\overline{z}) - \overline{z}(1-|w|^2). \tag{3.5}$$

Simple computations [20] demonstrate that

$$
\begin{aligned}
(3.6)\quad s(w,z)(w-z) &= \left\{\overline{w}\left(1-w\overline{z}\right)-\overline{z}\left(1-|w|^2\right)\right\}(w-z)\\
&= \left\{\left(\overline{w}-\overline{z}\right)-\left(w\overline{z}\right)\overline{w}+|w|^2\,\overline{z}\right\}(w-z)\\
&= |w-z|^2-\left(w\overline{z}\right)\left(|w|^2-\overline{w}z\right)+|w|^2\left(\overline{z}w-|z|^2\right)\\
&= |w-z|^2-\left(w\overline{z}\right)|w|^2+|\overline{w}z|^2+|w|^2\,\overline{z}w-|w|^2\,|z|^2\\
&= |w-z|^2+|\overline{w}z|^2-|w|^2\,|z|^2=\triangle\left(w,z\right),
\end{aligned}
$$

by the second line in (3.1).

Definition 1. *We define the Cauchy Kernel on $\mathbb{B}_n\times\mathbb{B}_n$ to be*

$$\mathcal{C}_n\left(w,z\right)\equiv s^*\mu(w,z) \tag{3.7}$$

for the section s given in (3.5) above.

Definition 2. *For $0\leq p\leq n$ and $0\leq q\leq n-1$ we let $\mathcal{C}_n^{p,q}$ be the component of $\mathcal{C}_n\left(w,z\right)$ that has bidegree (p,q) in z and bidegree $(n-p,n-q-1)$ in w.*

Thus if η is a $(p,q+1)$-form in w, then $\mathcal{C}_n^{p,q}\wedge\eta$ is a (p,q)-form in z and a multiple of the volume form in w. We now prepare to give explicit formulas for Charpentier's solution kernels $\mathcal{C}_n^{0,q}(w,z)$. First we introduce some notation.

Notation 2. *Let $\omega_n\left(z\right)=\bigwedge_{j=1}^n dz_j$. For n a positive integer and $0\leq q\leq n-1$ let P_n^q denote the collection of all permutations ν on $\{1,\dots,n\}$ that map to $\{i_\nu,J_\nu,L_\nu\}$ where J_ν is an increasing multi-index with $card(J_\nu)=n-q-1$ and $card(L_\nu)=q$. Let $\epsilon_\nu\equiv sgn\left(\nu\right)\in\{-1,1\}$ denote the signature of the permutation ν.*

Note that the number of increasing multi-indices of length $n-q-1$ is $\frac{n!}{(q+1)!(n-q-1)!}$, while the number of increasing multi-indices of length q are $\frac{n!}{q!(n-q)!}$. Since we are only allowed certain combinations of J_ν and L_ν (they must have disjoint intersection and they must be increasing multi-indices), it is straightforward to see that the total number of permutations in P_n^q that we are considering is $\frac{n!}{(n-q-1)!q!}$.

From Øvrelid [22] we obtain that Charpentier's kernel takes the (abstract) form

$$\mathcal{C}_n^{0,q}(w,z)=\frac{1}{\triangle(w,z)^n}\sum_{\nu\in P_n^q}sgn\left(\nu\right)s_{i_\nu}\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j\bigwedge_{l\in L_\nu}\overline{\partial}_z s_l\wedge\omega_n(w).$$

Fundamental for us will be the explicit formula for Charpentier's kernel given in the next theorem. We are informed by Part 2 of Proposition I.1 in [13] that $\mathcal{C}_n^{p,q}(w,z)=0$ for $w\in\partial\mathbb{B}_n$, and this serves as a guiding principle in the proof we give in the appendix. It is convenient to isolate the following factor common to all summands in the formula:

$$\Phi_n^q\left(w,z\right)\equiv\frac{\left(1-w\overline{z}\right)^{n-1-q}\left(1-|w|^2\right)^q}{\triangle\left(w,z\right)^n},\qquad 0\leq q\leq n-1. \tag{3.8}$$

Theorem 4. *Let n be a positive integer and suppose that $0\leq q\leq n-1$. Then*

$$\mathcal{C}_n^{0,q}\left(w,z\right)=\sum_{\nu\in P_n^q}\left(-1\right)^q\Phi_n^q\left(w,z\right)sgn\left(\nu\right)\left(\overline{w_{i_\nu}}-\overline{z_{i_\nu}}\right)\bigwedge_{j\in J_\nu}d\overline{w_j}\bigwedge_{l\in L_\nu}d\overline{z_l}\bigwedge\omega_n\left(w\right). \tag{3.9}$$

Remark 2. *We can rewrite the formula for $\mathcal{C}_n^{0,q}(w,z)$ in (3.9) as*

(3.10)
$$\mathcal{C}_n^{0,q}(w,z) = \Phi_n^q(w,z) \sum_{|J|=q} \sum_{k\notin J} (-1)^{\mu(k,J)} \left(\overline{z_k} - \overline{w_k}\right) d\overline{z}^J \wedge d\overline{w}^{(J\cup\{k\})^c} \wedge \omega_n(w),$$

where $J\cup\{k\}$ here denotes the increasing multi-index obtained by rearranging the integers $\{k, j_1, ... j_q\}$ as

$$J\cup\{k\} = \left\{j_1, ... j_{\mu(k,J)-1}, k, j_{\mu(k,J)}, ... j_q\right\}.$$

Thus k occupies the $\mu(k,J)^{th}$ position in $J\cup\{k\}$. The notation $(J\cup\{k\})^c$ refers to the increasing multi-index obtained by rearranging the integers in $\{1,2,...n\}\setminus(J\cup\{k\})$. To see (3.10), we note that in (3.9) the permutation ν takes the n-tuple $(1,2,...n)$ to (i_ν, J_ν, L_ν). In (3.10) the n-tuple $(k,(J\cup\{k\})^c, J)$ corresponds to (i_ν, J_ν, L_ν), and so sgn(ν) becomes in (3.10) the signature of the permutation that takes $(1,2,...n)$ to $(k,(J\cup\{k\})^c, J)$. This in turn equals $(-1)^{\mu(k,J)}$ with $\mu(k,J)$ as above.

We observe at this point that the functional coefficient in the summands in (3.9) looks like

$$(-1)^q \Phi_n^q(w,z)\left(\overline{w_{i_\nu}} - \overline{z_{i_\nu}}\right) = (-1)^q \frac{(1-w\overline{z})^{n-q-1}(1-|w|^2)^q}{\triangle(w,z)^n}\left(\overline{w_{i_\nu}} - \overline{z_{i_\nu}}\right),$$

which behaves like a fractional integral operator of order 1 in the Bergman metric on the diagonal relative to invariant measure. See the appendix for a proof of Theorem 4.

Finally, we will adopt the usual convention of writing

$$\mathcal{C}_n^{0,q} f(z) = \int_{\mathbb{B}_n} f(w) \wedge \mathcal{C}_n^{0,q}(w,z),$$

when we wish to view $\mathcal{C}_n^{0,q}$ as an operator taking $(0,q+1)$-forms f in w to $(0,q)$-forms $\mathcal{C}_n^{0,q} f$ in z.

3.1. **Ameliorated kernels.** We now wish to define right inverses with improved behaviour at the boundary. We consider the case when the right side f of the $\overline{\partial}$ equation is a $(p,q+1)$-form in $\mathbb{B}_n$.

As usual for a positive integer $s>n$ we will "project" the formula $\overline{\partial}\mathcal{C}_s^{p,q} f = f$ in $\mathbb{B}_s$ for a $\overline{\partial}$-closed form f in $\mathbb{B}_s$ to a formula $\overline{\partial}\mathcal{C}_{n,s}^{p,q} f = f$ in $\mathbb{B}_n$ for a $\overline{\partial}$-closed form f in $\mathbb{B}_n$. To accomplish this we define *ameliorated* operators $\mathcal{C}_{n,s}^{p,q}$ by

$$\mathcal{C}_{n,s}^{p,q} = \mathsf{R}_n \mathcal{C}_s^{p,q} \mathsf{E}_s,$$

where for $n<s$, E_s (R_n) is the extension (restriction) operator that takes forms $\Omega = \sum \eta_{I,J} dw^I \wedge d\overline{w}^J$ in $\mathbb{B}_n$ ($\mathbb{B}_s$) and extends (restricts) them to $\mathbb{B}_s$ ($\mathbb{B}_n$) by

$$\begin{aligned} \mathsf{E}_s\left(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J\right) &\equiv \sum \left(\eta_{I,J}\circ R\right) dw^I \wedge d\overline{w}^J, \\ \mathsf{R}_n\left(\sum \eta_{I,J} dw^I \wedge d\overline{w}^J\right) &\equiv \sum_{I,J\subset\{1,2,...,n\}} \left(\eta_{I,J}\circ E\right) dw^I \wedge d\overline{w}^J. \end{aligned}$$

Here R is the natural orthogonal projection from $\mathbb{C}^s$ to $\mathbb{C}^n$ and E is the natural embedding of $\mathbb{C}^n$ into $\mathbb{C}^s$. In other words, we extend a form by taking the coefficients to be constant in the extra variables, and we restrict a form by discarding all wedge

products of differentials involving the extra variables and restricting the coefficients accordingly.

For $s>n$ we observe that the operator $\mathcal{C}_{n,s}^{p,q}$ has integral kernel

$$\mathcal{C}_{n,s}^{p,q}(w,z)\equiv\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}}\mathcal{C}_{s}^{p,q}\left((w,w'),(z,0)\right)dV(w'),\qquad z,w\in\mathbb{B}_n, \tag{3.11}$$

where $\mathbb{B}_{s-n}$ denotes the unit ball in $\mathbb{C}^{s-n}$ with respect to the orthogonal decomposition $\mathbb{C}^s=\mathbb{C}^n\oplus\mathbb{C}^{s-n}$, and dV denotes Lebesgue measure. If $f(w)$ is a $\overline{\partial}$-closed form on $\mathbb{B}_n$ then $f(w,w')=f(w)$ is a $\overline{\partial}$-closed form on $\mathbb{B}_s$ and we have for $z\in\mathbb{B}_n$,

$$\begin{aligned} f(z) &= f(z,0)=\overline{\partial}\int_{\mathbb{B}_s}\mathcal{C}_s^{p,q}\left((w,w'),(z,0)\right)f(w)\,dV(w)\,dV(w') \\ &= \overline{\partial}\int_{\mathbb{B}_n}\left\{\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}}\mathcal{C}_s^{p,q}\left((w,w'),(z,0)\right)dV(w')\right\}f(w)\,dV(w) \\ &= \overline{\partial}\int_{\mathbb{B}_n}\mathcal{C}_{n,s}^{p,q}(w,z)\,f(w)\,dV(w). \end{aligned}$$

We have proved that

$$\mathcal{C}_{n,s}^{p,q}f(z)\equiv\int_{\mathbb{B}_n}\mathcal{C}_{n,s}^{p,q}(w,z)\,f(w)\,dV(w)$$

is a right inverse for $\overline{\partial}$ on $\overline{\partial}$-closed forms:

Theorem 5. *For all $s>n$ and $\overline{\partial}$-closed forms f in $\mathbb{B}_n$, we have*

$$\overline{\partial}\mathcal{C}_{n,s}^{p,q}f=f\ \text{ in } \mathbb{B}_n.$$

We will use only the case $p=0$ of this theorem and from now on we restrict our attention to this case. The operators $\mathcal{C}_{n,s}^{0,0}$ have been computed in [20] and are given by

$$\mathcal{C}_{n,s}^{0,0}f(z)=\int_{\mathbb{B}_n}\sum_{j=0}^{n-1}c_{n,j,s}\frac{\left(1-|w|^2\right)^{s-n+j}\left(1-|z|^2\right)^{j}}{\left(1-\overline{w}z\right)^{s-n+j}\left(1-w\overline{z}\right)^{j}}\mathcal{C}_n^{0,0}(w,z)\wedge f(w), \tag{3.12}$$

where

$$\begin{aligned} \mathcal{C}_n^{0,0}(w,z) &= c_0\frac{\left(1-w\overline{z}\right)^{n-1}}{\left\{|1-w\overline{z}|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)\right\}^n} \\ &\quad\times\sum_{j=1}^{n}(-1)^{j-1}\left(\overline{w_j}-\overline{z_j}\right)\bigwedge_{k\neq j}d\overline{w_k}\bigwedge_{\ell=1}^{n}dw_\ell. \end{aligned}$$

A similar result holds for the operators $\mathcal{C}^{0,q}_{n,s}$. Define

$$\begin{aligned}
\Phi^q_{n,s}(w,z) &= \Phi^q_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^j \\
&= \frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^j \\
&= \sum_{j=0}^{n-q-1} c_{j,n,s}\frac{(1-w\overline{z})^{n-1-q-j}\left(1-|w|^2\right)^{s-n+q+j}\left(1-|z|^2\right)^j}{(1-\overline{w}z)^{s-n+j}\,\triangle(w,z)^n}.
\end{aligned}$$

Note that the numerator and denominator are *balanced* in the sense that the sum of the exponents in the denominator minus the corresponding sum in the numerator (counting $\triangle(w,z)$ double) is $s+n+j-(s+j-1)=n+1$, the exponent of the invariant measure of the ball $\mathbb{B}_n$.

Theorem 6. *Suppose that $s>n$ and $0\le q\le n-1$. Then we have*

$$\begin{aligned}
\mathcal{C}^{0,q}_{n,s}(w,z) &= \mathcal{C}^{0,q}_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^j \\
&= \Phi^q_{n,s}(w,z)\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\overline{z_k}-\overline{w_k}\right)d\overline{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w).
\end{aligned}$$

Proof: For $s>n$ recall that the kernels of the ameliorated operators $\mathcal{C}^{0,q}_{n,s}$ are given in (3.11). For ease of notation, we will set $k=s-n$, so we have $\mathbb{C}^s=\mathbb{C}^n\oplus\mathbb{C}^k$. Suppose that $0\le q\le n-1$. Recall from (3.9) that

$$\begin{aligned}
\mathcal{C}^{0,q}_s(w,z) &= (-1)^q\frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^q}{\triangle(w,z)^s} \\
&\quad\times\sum_{\nu\in P^q_s} sgn(\nu)\left(\overline{w_{i_\nu}}-\overline{z_{i_\nu}}\right)\bigwedge_{j\in J_\nu} d\overline{w_j}\bigwedge_{l\in L_\nu} d\overline{z_l}\bigwedge\omega_s(w) \\
&= \sum_{\nu\in P^q_s} F^q_{s,i_\nu}(w,z)\bigwedge_{j\in J_\nu} d\overline{w_j}\bigwedge_{l\in L_\nu} d\overline{z_l}\bigwedge\omega_s(w).
\end{aligned}$$

where

$$F^q_{s,i_\nu}(w,z)=\Phi^q_s(w,z)\left(\overline{w_{i_\nu}}-\overline{z_{i_\nu}}\right)=\frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^q}{\triangle(w,z)^s}\left(\overline{w_{i_\nu}}-\overline{z_{i_\nu}}\right).$$

To compute the ameliorations of these kernels, we need only focus on the functional coefficient $F^q_{s,i_\nu}(w,z)$ of the kernel. It is easy to see that the ameliorated kernel can only give a contribution in the variables when $1\le i_\nu\le n$, since when $n+1\le i_\nu\le s$ the functional kernel becomes radial in certain variables and thus reduces to zero upon integration.

Then for any $1 \leq i \leq n$ the corresponding functional coefficient $F_{s,i}^{q}(w,z)$ has amelioration $F_{n,s,i}^{q}(w,z)$ given by

$$\begin{aligned} F_{n,s,i}^{q}(w,z) &= \int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} F_{s,i}^{q}((w,w'),(z,0))\,dV(w') \\ &= \int_{\sqrt{1-|w|^2}\mathbb{B}_k} \frac{(1-w\overline{z})^{s-q-1}(1-|w|^2-|w'|^2)^q(\overline{z}_i-\overline{w}_i)}{\triangle((w,w'),(z,0))^s} dV(w') \\ &= (\overline{z}_i-\overline{w}_i)(1-w\overline{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_k} \frac{(1-|w|^2-|w'|^2)^q}{\triangle((w,w'),(z,0))^s} dV(w'). \end{aligned}$$

Theorem 6 is a thus a consequence of the following elementary lemma, which will find application in Section 4 below on integration by parts as well.

Lemma 1. *We have*

$$\begin{aligned} &(1-w\overline{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_{s-n}} \frac{(1-|w|^2-|w'|^2)^q}{\triangle((w,w'),(z,0))^s} dV(w') \\ = &\frac{\pi^{s-n}}{(s-n)!}\Phi_n^q(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-q-1} c_{j,n,s}\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\right)^j. \end{aligned}$$

See the appendix for a proof of Lemma 1.

4. Integration by parts

We begin with an integration by parts formula involving a covariant derivative in [20] (Lemma 2.1 on page 57) that reduces the singularity of the solution kernel on the diagonal at the expense of differentiating the form. However, in order to prepare for a generalization to higher order forms, we replace the covariant derivative with the notion of $\overline{\mathcal{Z}_{z,w}}$-derivative defined in (4.2) below.

Recall Charpentier's explicit solution $\mathcal{C}_n^{0,0}\eta$ to the $\overline{\partial}$ equation $\overline{\partial}\mathcal{C}_n^{0,0}\eta = \eta$ in the ball $\mathbb{B}_n$ when η is a $\overline{\partial}$-closed $(0,1)$-form with coefficients in $C(\overline{\mathbb{B}_n})$: the kernel is given by

$$\mathcal{C}_n^{0,0}(w,z) = c_0\frac{(1-w\overline{z})^{n-1}}{\triangle(w,z)^n}\sum_{j=1}^n (-1)^{j-1}(\overline{w_j}-\overline{z_j})\bigwedge_{k\neq j} d\overline{w_k}\bigwedge_{\ell=1}^n dw_\ell,$$

for $(w,z)\in\mathbb{B}_n\times\mathbb{B}_n$ where

$$\triangle(w,z) = |1-w\overline{z}|^2 - \left(1-|w|^2\right)\left(1-|z|^2\right).$$

Define the Cauchy operator $\mathcal{S}_n$ on $\partial\mathbb{B}_n\times\mathbb{B}_n$ with kernel

$$\mathcal{S}_n(\zeta,z) = c_1\frac{1}{\left(1-\overline{\zeta}z\right)^n}d\sigma(\zeta), \qquad (\zeta,z)\in\partial\mathbb{B}_n\times\mathbb{B}_n.$$

Let $\eta = \sum_{j=1}^n \eta_j d\overline{w_j}$ be a $(0,1)$-form with smooth coefficients. Let $\overline{\mathcal{Z}} = \overline{\mathcal{Z}_{z,w}}$ be the vector field acting in the variable $w=(w_1,...,w_n)$ and parameterized by $z=(z_1,...,z_n)$ given by

$$\overline{\mathcal{Z}} = \overline{\mathcal{Z}_{z,w}} = \sum_{j=1}^n (\overline{w_j}-\overline{z_j})\frac{\partial}{\partial\overline{w_j}}. \tag{4.1}$$

It will usually be understood from the context what the acting variable w and the parameter variable z are in $\overline{\mathcal{Z}_{z,w}}$ and we will then omit the subscripts and simply write $\overline{\mathcal{Z}}$ for $\overline{\mathcal{Z}_{z,w}}$.

Definition 3. *For $m \geq 0$, define the m^{th} order derivative $\overline{\mathcal{Z}}^m \eta$ of a $(0,1)$-form $\eta = \sum_{k=1}^n \eta_k(w)\, d\overline{w_k}$ to be the $(0,1)$-form obtained by componentwise differentiation holding monomials in $\overline{w} - \overline{z}$ fixed:*

$$\overline{\mathcal{Z}}^m \eta(w) = \sum_{k=1}^n \left(\overline{\mathcal{Z}}^m \eta_k\right)(w)\, d\overline{w_k} = \sum_{k=1}^n \left\{ \sum_{|\alpha|=m} \left(\overline{w-z}\right)^\alpha \frac{\partial^m \eta_k}{\partial \overline{w}^\alpha}(w) \right\} d\overline{w_k}. \tag{4.2}$$

Lemma 2. *(cf. Lemma 2.1 of [20]) For all $m \geq 0$ and smooth $(0,1)$-forms $\eta = \sum_{k=1}^n \eta_k(w)\, d\overline{w_k}$, we have the formula,*

$$\begin{aligned} \mathcal{C}_n^{0,0}\eta(z) &\equiv \int_{\mathbb{B}_n} \mathcal{C}_n^{0,0}(w,z) \wedge \eta(w) \\ &= \sum_{j=0}^{m-1} c_j \int_{\partial\mathbb{B}_n} \mathcal{S}_n(w,z) \left(\overline{\mathcal{Z}}^j \eta\right) \left[\overline{\mathcal{Z}}\right](w)\, d\sigma(w) \\ &\quad + c_m \int_{\mathbb{B}_n} \mathcal{C}_n^{0,0}(w,z) \wedge \overline{\mathcal{Z}}^m \eta(w). \end{aligned} \tag{4.3}$$

Here the $(0,1)$-form $\overline{\mathcal{Z}}^j \eta$ acts on the vector field $\overline{\mathcal{Z}}$ in the usual way:

$$\left(\overline{\mathcal{Z}}^j \eta\right)\left[\overline{\mathcal{Z}}\right] = \left(\sum_{k=1}^n \overline{\mathcal{Z}}^j \eta_k(w)\, d\overline{w_k}\right) \left(\sum_{i=1}^n \left(\overline{w_i} - \overline{z_i}\right) \frac{\partial}{\partial \overline{w_i}}\right) = \sum_{k=1}^n \left(\overline{w_k} - \overline{z_k}\right) \overline{\mathcal{Z}}^j \eta_k(w).$$

We can also rewrite the final integral in (4.3) as

$$\int_{\mathbb{B}_n} \mathcal{C}_n^{0,0}(w,z) \wedge \overline{\mathcal{Z}}^m \eta(w) = \int_{\mathbb{B}_n} \Phi_n^0(w,z) \left(\overline{\mathcal{Z}}^m \eta\right) \left[\overline{\mathcal{Z}}\right](w)\, dV(w).$$

See the appendix for a proof of Lemma 2.

We now extend Lemma 2 to $(0,q+1)$-forms. Let

$$\eta = \sum_{|I|=q+1} \eta_I(w)\, d\overline{w}^I$$

be a $(0,q+1)$-form with smooth coefficients. Given a $(0,q+1)$-form $\eta = \sum_{|I|=q+1} \eta_I d\overline{w}^I$ and an increasing sequence J of length $|J| = q$, we define the interior product $\eta \lrcorner d\overline{w}^J$ of η and $d\overline{w}^J$ by

$$\eta \lrcorner d\overline{w}^J = \sum_{|I|=q+1} \eta_I d\overline{w}^I \lrcorner d\overline{w}^J = \sum_{k \notin J} (-1)^{\mu(k,J)} \eta_{J \cup \{k\}} d\overline{w_k}, \tag{4.4}$$

since $d\overline{w}^I \lrcorner d\overline{w}^J = (-1)^{\mu(k,J)} d\overline{w_k}$ if $k \in I \setminus J$ is the $\mu(k,J)^{th}$ index in I, and 0 otherwise. Recall the vector field $\overline{\mathcal{Z}}$ defined in (4.1). The key connection between

$\eta \lrcorner d\overline{w}^J$ and the vector field $\overline{\mathcal{Z}}$ is

$$\left(\eta \lrcorner d\overline{w}^J\right)\left(\overline{\mathcal{Z}}\right) = \left(\sum_{k=1}^{n} (-1)^{\mu(k,J)} \eta_{J\cup\{k\}} d\overline{w_k}\right)\left(\sum_{j=1}^{n} \left(\overline{w_j} - \overline{z_j}\right) \frac{\partial}{\partial \overline{w_j}}\right) = \sum_{k=1}^{n} \left(\overline{w_k} - \overline{z_k}\right)(-1)^{\mu(k,J)} \eta_{J\cup\{k\}}. \tag{4.5}$$

We now define an m^{th} order derivative $\overline{\mathcal{D}}^m \eta$ of a $(0, q+1)$-form η using the interior product. In the case $q = 0$ we will have $\overline{\mathcal{D}}^m \eta = \left(\overline{\mathcal{Z}}^m \eta\right)\left[\overline{\mathcal{Z}}\right]$ for a $(0,1)$-form η.

Remark 3. *We are motivated by the fact that the Charpentier kernel $\mathcal{C}_n^{0,q}(w, z)$ takes $(0, q+1)$-forms in w to $(0, q)$-forms in z. Thus in order to express the solution operator $\mathcal{C}_n^{0,q}$ in terms of a volume integral rather than the integration of a form in w and z, our definition of $\overline{\mathcal{D}}^m \eta$, even when $m = 0$, must include an appropriate exchange of w-differentials for z-differentials.*

Definition 4. *Let $m \geq 0$. For a $(0, q+1)$-form $\eta = \sum_{|I|=q+1} \eta_I d\overline{w}^I$ in the variable w, define the $(0, q)$-form $\overline{\mathcal{D}}^m \eta$ in the variable z by*

$$\overline{\mathcal{D}}^m \eta(w) = \sum_{|J|=q} \overline{\mathcal{Z}}^m \left(\eta \lrcorner d\overline{w}^J\right)\left[\overline{\mathcal{Z}}\right](w)\, d\overline{z}^J.$$

Again it is usually understood what the acting and parameter variables are in $\overline{\mathcal{D}}^m$ but we will write $\overline{\mathcal{D}_{z,w}}^m \eta(w)$ when this may not be the case. Note that for a $(0, q+1)$-form $\eta = \sum_{|I|=q+1} \eta_I d\overline{w}^I$, we have

$$\eta = \sum_{|J|=q} \left(\eta \lrcorner d\overline{w}^J\right) \wedge d\overline{w}^J,$$

and using (4.2) the above definition yields

$$\begin{aligned}
&\overline{\mathcal{D}}^m \eta(w) \\
&= \sum_{|J|=q} \overline{\mathcal{Z}}^m \left(\eta \lrcorner d\overline{w}^J\right)\left[\overline{\mathcal{Z}}\right](w)\, d\overline{z}^J \\
&= \sum_{|J|=q} \sum_{k=1}^{n} \left(\overline{w_k} - \overline{z_k}\right)(-1)^{\mu(k,J)} \left(\overline{\mathcal{Z}}^m \eta_{J\cup\{k\}}\right)(w)\, d\overline{z}^J \\
&= \sum_{|J|=q} \sum_{k=1}^{n} \left(\overline{w_k} - \overline{z_k}\right)(-1)^{\mu(k,J)} \left\{\sum_{|\alpha|=m} \left(\overline{w - z}\right)^{\alpha} \frac{\partial^m \eta_{J\cup\{k\}}}{\partial \overline{w}^{\alpha}}(w)\right\} d\overline{z}^J.
\end{aligned} \tag{4.6}$$

Thus the effect of $\overline{\mathcal{D}}^m$ on a basis element $\eta_I d\overline{w}^I$ is to replace a differential $d\overline{w_k}$ from $d\overline{w}^I$ $(I = J \cup \{k\})$ with the factor $(-1)^{\mu(k,J)}\left(\overline{w_k} - \overline{z_k}\right)$ (and this is accomplished by acting a $(0,1)$-form on $\overline{\mathcal{Z}}$), replace the remaining differential $d\overline{w}^J$ with $d\overline{z}^J$, and then to apply the differential operator $\overline{\mathcal{Z}}^m$ to the coefficient η_I. We will refer to the factor $\left(\overline{w_k} - \overline{z_k}\right)$ introduced above as a *rogue* factor since it is not associated with a derivative $\frac{\partial}{\partial \overline{w_k}}$ in the way that $\left(\overline{w - z}\right)^{\alpha}$ is associated with $\frac{\partial^m}{\partial \overline{w}^{\alpha}}$. The point of this distinction will be explained in Section 8 on estimates for solution operators.

The following lemma expresses $\mathcal{C}_n^{0,q}\eta(z)$ in terms of integrals involving $\overline{\mathcal{D}}^j\eta$ for $0 \le j \le m$. Note that the overall effect is to reduce the singularity of the kernel on the diagonal by m factors of $\sqrt{\triangle(w,z)}$, at the cost of increasing by m the number of derivatives hitting the form η. Recall from (3.8) that

$$\Phi_n^\ell(w,z) \equiv \frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(w,z)^n}.$$

We define the operator Φ_n^ℓ on forms η by

$$\Phi_n^\ell\eta(z) = \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\eta(w)\,dV(w).$$

Lemma 3. *Let $q \ge 0$. For all $m \ge 0$ we have the formula,*

$$\mathcal{C}_n^{0,q}\eta(z) = \sum_{k=0}^{m-1} c_k\mathcal{S}_n\left(\overline{\mathcal{D}}^j\eta\right)(z) + \sum_{\ell=0}^{q} c_\ell\Phi_n^\ell\left(\overline{\mathcal{D}}^m\eta\right)(z). \tag{4.7}$$

The proof is simply a reprise of that of Lemma 2 complicated by the algebra that reduces matters to $(0,1)$-forms. See the appendix.

4.1. **The radial derivative.** Recall the radial derivative $R = \sum_{j=1}^n w_j\frac{\partial}{\partial w_j}$ from (6.4). Here is Lemma 2.2 on page 58 of [20]. See the appendix for a proof.

Lemma 4. *Let $b > -1$. For $\Psi \in C\left(\overline{\mathbb{B}_n}\right) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\begin{aligned} &\int_{\mathbb{B}_n}\left(1-|w|^2\right)^b\Psi(w)\,dV(w) \\ =\;&\int_{\mathbb{B}_n}\left(1-|w|^2\right)^{b+1}\left(\frac{n+b+1}{b+1}I + \frac{1}{b+1}R\right)\Psi(w)\,dV(w). \end{aligned}$$

Remark 4. *Typically the above lemma is applied with*

$$\Psi(w) = \frac{1}{(1-\overline{w}z)^s}\psi(w,z)$$

where z is a parameter in the ball $\mathbb{B}_n$ and

$$R\Psi(w) = \frac{1}{(1-\overline{w}z)^s}R\psi(w,z)$$

since $\frac{1}{(1-\overline{w}z)^s}$ is antiholomorphic in w.

We will also need to iterate Lemma 4, and for this purpose it is convenient to introduce for $m \ge 1$ the notation

$$\begin{aligned} R_b &= R_{b,n} = \frac{n+b+1}{b+1}I + \frac{1}{b+1}R, \\ R_b^m &= R_{b+m-1}R_{b+m-2}\ldots R_b = \prod_{k=1}^m R_{b+m-k}. \end{aligned}$$

Corollary 3. *Let $b > -1$. For $\Psi \in C\left(\overline{\mathbb{B}_n}\right) \cap C^\infty(\mathbb{B}_n)$ we have*

$$\begin{aligned} &\int_{\mathbb{B}_n}\left(1-|w|^2\right)^b\Psi(w)\,dV(w) \\ =\;&\int_{\mathbb{B}_n}\left(1-|w|^2\right)^{b+m}R_b^m\Psi(w)\,dV(w). \end{aligned}$$

Remark 5. *The important point in Corollary 3 is that combinations of radial derivatives R and the identity I are played off against powers of $1-|w|^2$. It will sometimes be convenient to write this identity as*

$$\int_{\mathbb{B}_n} F(w)\,dV(w) = \int_{\mathbb{B}_n} \mathcal{R}_b^m F(w)\,dV(w)$$

where

$$\mathcal{R}_b^m \equiv \left(1-|w|^2\right)^{b+m} R_b^m \left(1-|w|^2\right)^{-b}, \tag{4.8}$$

and provided that $\Psi(w) = \left(1-|w|^2\right)^{-b} F(w)$ lies in $C\left(\overline{\mathbb{B}_n}\right) \cap C^\infty(\mathbb{B}_n)$.

4.2. **Integration by parts in ameliorated kernels.** We must now extend Lemma 3 and Corollary 3 to the ameliorated kernels $\mathcal{C}_{n,s}^{0,q}$ given by

$$\mathcal{C}_{n,s}^{0,q} = \mathsf{R}_n \mathcal{C}_s^{0,q} \mathsf{E}_s.$$

Since Corollary 3 already applies to very general functions $\Psi(w)$, we need only consider an extension of Lemma 3. The procedure for doing this is to apply Lemma 3 to $\mathcal{C}_s^{0,q}$ in s dimensions, and then integrate out the additional variables using Lemma 1.

Lemma 5. *Suppose that $s > n$ and $0 \le q \le n-1$. For all $m \ge 0$ and smooth $(0,q+1)$-forms η in $\overline{\mathbb{B}_n}$ we have the formula,*

$$\mathcal{C}_{n,s}^{0,q}\eta(z) = \sum_{k=0}^{m-1} c'_{k,n,s} \mathcal{S}_{n,s}\left(\overline{\mathcal{D}}^k \eta\right)\left[\overline{\mathcal{Z}}\right](z) + \sum_{\ell=0}^{q} c_{\ell,n,s} \Phi_{n,s}^\ell \left(\overline{\mathcal{D}}^m \eta\right)(z),$$

where the ameliorated operators $\mathcal{S}_{n,s}$ and $\Phi_{n,s}^\ell$ have kernels given by,

$$\begin{aligned} \mathcal{S}_{n,s}(w,z) &= c_{n,s}\frac{\left(1-|w|^2\right)^{s-n-1}}{(1-\overline{w}z)^s} = c_{n,s}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n-1}\frac{1}{(1-\overline{w}z)^{n+1}}, \\ \Phi_{n,s}^\ell(w,z) &= \Phi_n^\ell(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\sum_{j=0}^{n-\ell-1} c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^j. \end{aligned}$$

Proof: Recall that for a smooth $(0,q+1)$-form $\eta(w) = \sum_{|I|=q+1}\eta_I d\overline{w}^I$ in $\overline{\mathbb{B}_n}$, the $(0,q)$-form $\overline{\mathcal{D}^m}\mathsf{E}_s\eta$ is given by

$$\begin{aligned} \overline{\mathcal{D}^m}\mathsf{E}_s\eta(w) &= \sum_{|J|=q}\overline{\mathcal{D}^m}\left(\eta \lrcorner d\overline{w}^J\right)d\overline{z}^J = \sum_{|J|=q}\overline{\mathcal{D}^m}\left(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w_k}\right)d\overline{z}^J \\ &= \sum_{|J|=q}\overline{\mathcal{D}^m}\left(\sum_{k\notin J}(-1)^{\mu(k,J)}\eta_{J\cup\{k\}}(w)\,d\overline{w_k}\right)d\overline{z}^J \\ &= \sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\sum_{|\alpha|=m}\overline{(w_k-z_k)(w-z)^\alpha}\frac{\partial^m}{\partial\overline{w}^\alpha}\eta_{J\cup\{k\}}(w)\right), \end{aligned}$$

where $J\cup\{k\}$ is a multi-index with entries in $\mathfrak{I}_n \equiv \{1,2,...,n\}$ since the coefficient η_I vanishes if I is not contained in $\mathfrak{I}_n$. Moreover, the multi-index α lies in $(\mathfrak{I}_n)^m$ since the coefficients η_I are constant in the variable $w' = (w_{n+1},...,w_s)$. Thus

$$\overline{\mathcal{D}^m_{(z,0),(w,w')}}\mathsf{E}_s\eta = \overline{\mathcal{D}^m_{z,w}}\eta = \overline{\mathcal{D}^m}\eta,$$

and we compute that

$$\begin{aligned}
& \mathsf{R}_n \Phi_s^\ell \left(\overline{\mathcal{D}^m_{(z,0),(w,w')}} \mathsf{E}_s \eta \right)(z) \\
= \ & \Phi_s^\ell \left(\overline{\mathcal{D}^m} \eta \right)((z,0)) \\
= \ & \sum_{|J|=q} \sum_{k \in \mathfrak{I}_n \setminus J} (-1)^{\mu(k,J)} \sum_{|\alpha|=m} \Phi_s^\ell \left(\overline{(w_k - z_k)} \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J \cup \{k\}} ((w,w')) \right)((z,0)),
\end{aligned}$$

where $J \cup \{k\} \subset \mathfrak{I}_n$ and $\alpha \in (\mathfrak{I}_n)^m$ and

$$\begin{aligned}
& \Phi_s^\ell \left(\overline{(w_k - z_k)} \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J \cup \{k\}} (w) \right)((z,0)) \\
= \ & \int_{\mathbb{B}_s} \frac{(1 - w\overline{z})^{s-1-\ell} \left(1 - |w|^2 - |w'|^2\right)^\ell}{\triangle((w,w'),(z,0))^s} \overline{(w_k - z_k)} \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J \cup \{k\}} (w)\, dV((w,w')) \\
= \ & \int_{\mathbb{B}_n} \left\{ (1 - w\overline{z})^{s-\ell-1} \int_{\mathbb{B}_{s-n}} \frac{\left(1 - |w|^2 - |w'|^2\right)^\ell}{\triangle((w,w'),(z,0))^s} dV(w') \right\} \\
& \times \overline{(w_k - z_k)} \overline{(w-z)^\alpha} \frac{\partial^m}{\partial \overline{w}^\alpha} \eta_{J \cup \{k\}} (w)\, dV(w).
\end{aligned}$$

By Lemma 1 the term in braces above equals

$$\frac{\pi^{s-n}}{(s-n)!} \Phi_n^\ell (w,z) \left(\frac{1 - |w|^2}{1 - \overline{w} z} \right)^{s-n} \sum_{j=0}^{n-\ell-1} c_{j,n,s} \left(\frac{\left(1 - |w|^2\right)\left(1 - |z|^2\right)}{|1 - w\overline{z}|^2} \right)^j,$$

and now performing the sum $\sum_{|J|=q} \sum_{k \in \mathfrak{I}_n \setminus J} (-1)^{\mu(k,J)} \sum_{|\alpha|=m}$ yields

$$\mathsf{R}_n \Phi_s^\ell \left(\overline{\mathcal{D}^m_{(z,0)}} \mathsf{E}_s \eta \right)(z) = \Phi_s^\ell \left(\overline{\mathcal{D}^m_z} \eta \right)((z,0)) = \Phi_{n,s}^\ell \left(\overline{\mathcal{D}^m_z} \eta \right)(z). \tag{4.9}$$

An even easier calculation using formula (1) in 1.4.4 on page 14 of [24] shows that

$$\mathsf{R}_n \mathcal{S}_s \left(\mathsf{E}_s \overline{\mathcal{D}^k_z} \eta \right)((z,0)) = \mathcal{S}_s \left(\overline{\mathcal{D}^k_z} \eta \right)((z,0)) = \mathcal{S}_{n,s} \left(\overline{\mathcal{D}^k_z} \eta \right)(z), \tag{4.10}$$

and now the conclusion of Lemma 5 follows from (4.9), (4.10), the definition $\mathcal{C}^{0,q}_{n,s} = \mathsf{R}_n \mathcal{C}^{0,q}_s \mathsf{E}_s$, and Lemma 3.

5. The Koszul complex

Here we briefly review the algebra behind the Koszul complex as presented for example in [18] in the finite dimensional setting. A more detailed treatment in that setting can be found in Section 5.5.3 of [25]. Fix h holomorphic as in (2.5). Now if $g = (g_j)_{j=1}^\infty$ satisfies $|g|^2 = \sum_{j=1}^\infty |g_j|^2 \geq \delta^2 > 0$, let

$$\Omega_0^1 = \frac{\overline{g}}{|g|^2} = \left(\frac{\overline{g_j}}{|g|^2} \right)_{j=1}^\infty = \left(\Omega_0^1 (j) \right)_{j=1}^\infty,$$

which we view as a 1-tensor (in $\ell^2 = \mathbb{C}^\infty$) of $(0,0)$-forms with components $\Omega_0^1(j) = \frac{\overline{g_j}}{|g|^2}$. Then $f = \Omega_0^1 h$ satisfies $\mathcal{M}_g f = f \cdot g = h$, but in general fails to be holomorphic. The Koszul complex provides a scheme which we now recall for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_g \Gamma_0^2$ that when subtracted from f above yields a holomorphic solution to the second line in (2.5). See below.

The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0 = \left(\overline{\partial}\frac{\overline{g_j}}{|g|^2}\right)_{j=1}^{\infty} = \left(\overline{\partial}\Omega_0^1\left(j\right)\right)_{j=1}^{\infty}$ is given by

$$\overline{\partial}\Omega_0^1\left(j\right) = \overline{\partial}\frac{\overline{g_j}}{|g|^2} = \frac{|g|^2\,\overline{\partial g_j} - \overline{g_j}\overline{\partial}\,|g|^2}{|g|^4} = \frac{1}{|g|^4}\sum_{k=1}^{\infty} g_k\overline{\{g_k\partial g_j - g_j\partial g_k\}}.$$

and can be written as

$$\overline{\partial}\Omega_0^1 = \Lambda_g\Omega_1^2 \equiv \left[\sum_{k=1}^{\infty}\Omega_1^2\left(j,k\right)g_k\right]_{j=1}^{\infty},$$

where the antisymmetric 2-tensor Ω_1^2 of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2\left(j,k\right)\right]_{j,k=1}^{\infty} = \left[\frac{\overline{\{g_k\partial g_j - g_j\partial g_k\}}}{|g|^4}\right]_{j,k=1}^{\infty}.$$

and $\Lambda_g\Omega_1^2$ denotes its contraction by the vector g in the final variable.

We can repeat this process and by induction we have

$$\overline{\partial}\Omega_q^{q+1} = \Lambda_g\Omega_{q+1}^{q+2}, \qquad 0 \leq q \leq n, \tag{5.1}$$

where Ω_q^{q+1} is an alternating $(q+1)$-tensor of $(0,q)$-forms. Recall that h is holomorphic. When $q = n$ we have that $\Omega_n^{n+1}h$ is $\overline{\partial}$-closed and this allows us to solve a chain of $\overline{\partial}$ equations

$$\overline{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q}h - \Lambda_g\Gamma_{q-1}^{q+1},$$

for alternating q-tensors Γ_{q-2}^{q} of $(0,q-2)$-forms, using the ameliorated Charpentier solution operators $\mathcal{C}_{n,s}^{0,q}$ defined in (3.11) above (note that our notation suppresses the dependence of Γ on h). With the convention that $\Gamma_n^{n+2} \equiv 0$ we have

$$\begin{aligned} \overline{\partial}\left(\Omega_q^{q+1}h - \Lambda_g\Gamma_q^{q+2}\right) &= 0, \qquad 0 \leq q \leq n, \\ \overline{\partial}\Gamma_{q-1}^{q+1} &= \Omega_q^{q+1}h - \Lambda_g\Gamma_q^{q+2}, \qquad 1 \leq q \leq n. \end{aligned} \tag{5.2}$$

Now

$$f \equiv \Omega_0^1 h - \Lambda_g\Gamma_0^2$$

is holomorphic by the first line in (5.2) with $q = 0$, and since Γ_0^2 is antisymmetric, we compute that $\Lambda_g\Gamma_0^2 \cdot g = \Gamma_0^2\left(g,g\right) = 0$ and

$$\mathcal{M}_g f = f \cdot g = \Omega_0^1 h \cdot g - \Lambda_g\Gamma_0^2 \cdot g = h - 0 = h.$$

Thus $f = \left(f_i\right)_{i=1}^{\infty}$ is a vector of holomorphic functions satisfying the second line in (2.5). The first line in (2.5) is the subject of the remaining sections of the paper.

5.1. **Wedge products and factorization of the Koszul complex.** Here we record the remarkable factorization of the Koszul complex in Andersson and Carlsson [4]. To describe the factorization we introduce an exterior algebra structure on $\ell^2 = \mathbb{C}^{\infty}$. Let $\{e_1, e_2, ...\}$ be the usual basis in $\mathbb{C}^{\infty}$, and for an increasing multiindex $I = (i_1, ..., i_\ell)$ of integers in $\mathbb{N}$, define

$$e_I = e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_\ell},$$

where we use $\wedge$ to denote the wedge product in the exterior algebra $\Lambda^*\left(\mathbb{C}^{\infty}\right)$ of $\mathbb{C}^{\infty}$, as well as for the wedge product on forms in $\mathbb{C}^n$. Note that $\{e_I : |I| = r\}$ is a basis for the alternating r-tensors on $\mathbb{C}^{\infty}$.

If $f = \sum_{|I|=r} f_I e_I$ is an alternating r-tensor on $\mathbb{C}^{\infty}$ with values that are $(0,k)$-forms in $\mathbb{C}^n$, which may be viewed as a member of the exterior algebra of $\mathbb{C}^{\infty} \otimes \mathbb{C}^n$,

and if $g = \sum_{|J|=s} g_J e_J$ is an alternating s-tensor on $\mathbb{C}^\infty$ with values that are $(0,\ell)$-forms in $\mathbb{C}^n$, then as in [4] we define the wedge product $f \wedge g$ in the exterior algebra of $\mathbb{C}^\infty \otimes \mathbb{C}^n$ to be the alternating $(r+s)$-tensor on $\mathbb{C}^\infty$ with values that are $(0,k+\ell)$-forms in $\mathbb{C}^n$ given by

$$\begin{aligned} f \wedge g &= \left(\sum_{|I|=r} f_I e_I \right) \wedge \left(\sum_{|J|=s} g_J e_J \right) \\ &= \sum_{|I|=r,|J|=s} (f_I \wedge g_J)(e_I \wedge e_J) \\ &= \sum_{|K|=r+s} \left(\pm \sum_{I+J=K} f_I \wedge g_J \right) e_K . \end{aligned} \tag{5.3}$$

Note that we simply write the exterior product of an element from $\Lambda^*(\mathbb{C}^\infty)$ with an element from $\Lambda^*(\mathbb{C}^n)$ as juxtaposition, without writing an explicit wedge symbol. This should cause no confusion since the basis we use in $\Lambda^*(\mathbb{C}^\infty)$ is $\{e_i\}_{i=1}^\infty$, while the basis we use in $\Lambda^*(\mathbb{C}^n)$ is $\{dz_j, d\widehat{z_j}\}_{j=1}^n$, quite different in both appearance and interpretation.

In terms of this notation we then have the following factorization in Theorem 3.1 of Andersson and Carlsson [4]:

$$\Omega_0^1 \wedge \bigwedge_{i=1}^{\ell} \widetilde{\Omega_0^1} = \left(\sum_{k_0=1}^\infty \frac{\overline{g_{k_0}}}{|g|^2} e_{k_0} \right) \wedge \bigwedge_{i=1}^{\ell} \left(\sum_{k_i=1}^\infty \frac{\overline{\partial g_{k_i}}}{|g|^2} e_{k_i} \right) = -\frac{1}{\ell+1} \Omega_\ell^{\ell+1}, \tag{5.4}$$

where

$$\Omega_0^1 = \left(\frac{\overline{g_i}}{|g|^2} \right)_{i=1}^\infty \text{ and } \widetilde{\Omega_0^1} = \left(\frac{\overline{\partial g_i}}{|g|^2} \right)_{i=1}^\infty .$$

The factorization in [4] is proved in the finite dimensional case, but this extends to the infinite dimensional case by continuity. Since the ℓ^2 norm is quasi-multiplicative on wedge products by Lemma 5.1 in [4] we have

$$\left| \Omega_\ell^{\ell+1} \right|^2 \le C_\ell \left| \Omega_0^1 \right|^2 \left| \widetilde{\Omega_0^1} \right|^{2\ell}, \qquad 0 \le \ell \le n, \tag{5.5}$$

where the constant C_ℓ depends only on the number of factors ℓ in the wedge product, and *not* on the underlying dimension of the vector space (which is infinite for $\ell^2 = \mathbb{C}^\infty$).

It will be useful in the next section to consider also tensor products

$$\widetilde{\Omega_0^1} \otimes \widetilde{\Omega_0^1} = \left(\sum_{i=1}^\infty \frac{\overline{\partial g_i}}{|g|^2} e_i \right) \otimes \left(\sum_{j=1}^\infty \frac{\overline{\partial g_j}}{|g|^2} e_j \right) = \sum_{i,j=1}^\infty \frac{\overline{\partial g_i} \otimes \overline{\partial g_j}}{|g|^4} e_i \otimes e_j, \tag{5.6}$$

and more generally $\mathcal{X}^\alpha \widetilde{\Omega_0^1} \otimes \mathcal{X}^\beta \widetilde{\Omega_0^1}$ where $\mathcal{X}^m$ denotes the vector derivative defined in Definition 7 below. We will use the fact that the ℓ^2-norm is *multiplicative* on tensor products.

6. An almost invariant holomorphic derivative

In this section we continue to consider ℓ^2-valued spaces. We refer the reader to [6] for the definition of the Bergman tree $\mathcal{T}_n$ and the corresponding pairwise disjoint decomposition of the ball $\mathbb{B}_n$:

$$\mathbb{B}_n = \dot{\bigcup_{\alpha \in \mathcal{T}_n}} K_\alpha,$$

where the sets K_α are comparable to balls of radius one in the Bergman metric β on the ball $\mathbb{B}_n$: $\beta(z,w) = \frac{1}{2} \ln \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$ (Proposition 1.21 in [36]). This decomposition gives an analogue in $\mathbb{B}_n$ of the standard decomposition of the upper half plane $\mathbb{C}_+$ into dyadic squares whose distance from the boundary $\partial\mathbb{C}_+$ equals their side length. We also recall from [6] the differential operator D_a which on the Bergman kube K_α, and provided $a \in K_\alpha$, is close to the invariant gradient $\widetilde{\nabla}$, and which has the additional property that $D_a^m f(z)$ is holomorphic for $m \geq 1$ and $z \in K_\alpha$ when f is holomorphic. For our purposes the powers $D_a^m f$, $m \geq 1$, are easier to work with than the corresponding powers $\widetilde{\nabla}^m f$, which fail to be holomorphic. It is shown in [6] that D_a^m can be used to define an equivalent norm on the Besov space $B_p(\mathbb{B}_n) = B_p^0(\mathbb{B}_n)$, and it is a routine matter to extend this result to the Besov-Sobolev space $B_p^\sigma(\mathbb{B}_n)$ when $\sigma \geq 0$ and $m > 2\left(\frac{n}{p} - \sigma\right)$. The further extension to ℓ^2 -valued functions is also routine.

We define

$$\nabla_z = \left(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\right) \text{ and } \overline{\nabla_z} = \left(\frac{\partial}{\partial \overline{z_1}}, ..., \frac{\partial}{\partial \overline{z_n}}\right)$$

so that the usual Euclidean gradient is given by the pair $\left(\nabla_z, \overline{\nabla_z}\right)$. Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha$. Recall that the gradient with invariant length given by

$$\begin{aligned} \widetilde{\nabla} f(a) &= (f \circ \varphi_a)'(0) = f'(a)\varphi_a'(0) \\ &= -f'(a)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\} \end{aligned}$$

fails to be holomorphic in a. To rectify this, we define as in [6],

$$\begin{aligned} D_a f(z) &= f'(z)\varphi_a'(0) \\ &= -f'(z)\left\{\left(1-|a|^2\right)P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}, \end{aligned} \tag{6.1}$$

for $z \in \mathbb{B}_n$. Note that $\nabla_z(\overline{a} \cdot z) = \overline{a}^t$ when we view $w \in \mathbb{B}_n$ as an $n \times 1$ complex matrix, and denote by w^t the $1 \times n$ transpose of w. With this interpretation, we observe that $P_a z = \frac{\overline{a} z}{|a|^2} a$ has derivative $P_a = P_a' z = \frac{a \overline{a}^t}{|a|^2} = |a|^{-2}\left[a_i \overline{a_j}\right]_{1 \leq i,j \leq n}$.

The next lemma from [6] shows that D_a^m and D_b^m are comparable when a and b are close in the Bergman metric.

Lemma 6. *(Lemma 6.2 in [6]) Let $a, b \in \mathbb{B}_n$ satisfy $\beta(a,b) \leq C$. There is a positive constant C_m depending only on C and m such that*

$$C_m^{-1}\left|D_b^m f(z)\right| \leq \left|D_a^m f(z)\right| \leq C_m \left|D_b^m f(z)\right|,$$

for all $f \in H\left(\mathbb{B}_n; \ell^2\right)$.

We remind the reader that $|D_a^m f(z)| = \sqrt{\sum_{i=1}^{\infty} |D_a^m f_i(z)|^2}$ if $f = (f_i)_{i=1}^{\infty}$. The scalar proof in [6] is easily extended to ℓ^2-valued f.

Definition 5. *(see [6]) Suppose $\sigma \geq 0$, $1 < p < \infty$ and $m \geq 1$. We define a "tree semi-norm" $\|\cdot\|^*_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ by*

$$\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} = \left(\sum_{\alpha \in \mathcal{T}_n} \int_{B_d(c_\alpha, C_2)} \left| \left(1 - |z|^2\right)^\sigma D^m_{c_\alpha} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}}. \tag{6.2}$$

We now recall the invertible "radial" operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [36] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. This definition is easily extended to $f \in H\left(\mathbb{B}_n;\ell^2\right)$. If the inverse of $R^{\gamma,t}$ is denoted $R_{\gamma,t}$, then Proposition 1.14 of [36] yields

$$\begin{aligned} R^{\gamma,t}\left(\frac{1}{(1-\overline{w}z)^{n+1+\gamma}}\right) &= \frac{1}{(1-\overline{w}z)^{n+1+\gamma+t}}, \\ R_{\gamma,t}\left(\frac{1}{(1-\overline{w}z)^{n+1+\gamma+t}}\right) &= \frac{1}{(1-\overline{w}z)^{n+1+\gamma}}, \end{aligned} \tag{6.3}$$

for all $w \in \mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t. The next proposition shows that the derivatives $R^{\gamma,m} f(z)$ are "L^p norm equivalent" to $\left\{f(0), \ldots, \nabla^{m-1} f(0), \nabla^m f(z)\right\}$ for m large enough. The scalar case $\sigma = 0$ is Proposition 2.1 in [6] and follows from Theorems 6.1 and Theorem 6.4 of [36]. The extension to $\sigma \geq 0$ and ℓ^2-valued f is routine. See the appendix and also [11].

Proposition 1. *Suppose that $\sigma \geq 0$, $0 < p < \infty$, $n+\gamma$ is not a negative integer, and $f \in H\left(\mathbb{B}_n;\ell^2\right)$. Then the following four conditions are equivalent:*

$$\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for some } m > \frac{n}{p} - \sigma, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for all } m > \frac{n}{p} - \sigma, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for some } m > \frac{n}{p} - \sigma, m+n+\gamma \notin -\mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for all } m > \frac{n}{p} - \sigma, m+n+\gamma \notin -\mathbb{N}.$$

Moreover, with $\psi(z) = 1 - |z|^2$, we have for $1 < p < \infty$,

$$\begin{aligned} & C^{-1} \left\|\psi^{m_1+\sigma} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n;\ell^2)} \\ \leq\ & \sum_{k=0}^{m_2-1} \left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m_2+\sigma} \nabla^{m_2} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} \\ \leq\ & C \left\|\psi^{m_1+\sigma} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n;\ell^2)} \end{aligned}$$

for all $m_1, m_2 > \frac{n}{p} - \sigma$, $m_1 + n + \gamma \notin -\mathbb{N}$, $m_2 \in \mathbb{N}$, *and where the constant* C *depends only on* σ, m_1, m_2, n, γ *and* p.

There is one further equivalent norm involving the radial derivative

$$Rf(z) = z \cdot \nabla f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z), \tag{6.4}$$

and its iterates $R^k = R \circ R \circ ... \circ R$ (k times).

Proposition 2. *Suppose that* $\sigma \geq 0$, $0 < p < \infty$ *and* $f \in H\left(\mathbb{B}_n; \ell^2\right)$. *Then*

$$\begin{aligned} &\sum_{k=0}^{m_1} \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m_1 + \sigma} R^k f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} \\ \approx &\sum_{k=0}^{m_2 - 1} \left| \nabla^k f(0) \right| + \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m_2 + \sigma} \nabla^{m_2} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} \end{aligned}$$

for all $m_1, m_2 > \frac{n}{p} - \sigma$, $m_1 + n + \gamma \notin -\mathbb{N}$, $m_2 \in \mathbb{N}$, *and where the constants in the equivalence depend only on* σ, m_1, m_2, n *and* p.

The seminorms $\|\cdot\|^*_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ turn out to be independent of $m > 2\left(\frac{n}{p} - \sigma\right)$. We will obtain this fact as a corollary of the equivalence of the standard norm in (2.4) with the corresponding norm in Proposition 1 using the "radial" derivative $R^{0,m}$. Note that the restriction $m > 2\left(\frac{n}{p} - \sigma\right)$ is dictated by the fact that $\left|D^m_{c_\alpha} f(z)\right|$ involves the factor $\left(1 - |z|^2\right)^{\frac{m}{2}}$ times m^{th} order tangential derivatives of f, and so we must have that $\left(1 - |z|^2\right)^{\left(\frac{m}{2} + \sigma\right)p} d\lambda_n(z)$ is a finite measure, i.e. $\left(\frac{m}{2} + \sigma\right)p - n - 1 > -1$. The case scalar $\sigma = 0$ of the following lemma is Lemma 6.4 in [6].

Lemma 7. *Let* $1 < p < \infty$, $\sigma \geq 0$ *and* $m > 2\left(\frac{n}{p} - \sigma\right)$. *Denote by* $B_\beta(c, C)$ *the ball center* c *radius* C *in the Bergman metric* β. *Then for* $f \in H\left(\mathbb{B}_n; \ell^2\right)$,

$$\begin{aligned} &\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} + \sum_{j=0}^{m-1} \left| \nabla^j f(0) \right| \\ \equiv &\left(\sum_{\alpha \in \mathcal{T}_n} \int_{B_\beta(c_\alpha, C_2)} \left| \left(1 - |z|^2\right)^\sigma D^m_{c_\alpha} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left| \nabla^j f(0) \right| \\ \approx &\left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m + \sigma} R^{\sigma, m} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left| \nabla^j f(0) \right| = \|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}. \end{aligned} \tag{6.5}$$

See the appendix for an adaptation of the proof in [6] to the case $\sigma \geq 0$ and ℓ^2-valued f.

We will also need to know that the pointwise multipliers in $M_{B^\sigma_p(\mathbb{B}_n) \to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ are bounded. Indeed, standard arguments show the following.

Lemma 8.

$$M_{B^\sigma_p(\mathbb{B}_n) \to B^\sigma_p(\mathbb{B}_n;\ell^2)} \subset H^\infty\left(\mathbb{B}_n; \ell^2\right) \cap B^\sigma_p\left(\mathbb{B}_n; \ell^2\right). \tag{6.6}$$

Proof: If $\varphi \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$, then $\varphi \in B_p^\sigma\left(\mathbb{B}_n;\ell^2\right)$ since $1 \in B_p^\sigma\left(\mathbb{B}_n\right)$, and

$$\mathbb{M}_\varphi : B_p^\sigma\left(\mathbb{B}_n\right) \to B_p^\sigma\left(\mathbb{B}_n;\ell^2\right) \text{ and } \mathbb{M}_\varphi^* : B_p^\sigma\left(\mathbb{B}_n;\ell^2\right)^* \to B_p^\sigma\left(\mathbb{B}_n\right)^*.$$

If e_z denotes point evaluation at $z \in \mathbb{B}_n$, $x \in \ell^2$ and $f \in B_p^\sigma\left(\mathbb{B}_n\right)$, then the calculation

$$\begin{aligned}
\left\langle f, \mathbb{M}_\varphi^*\left(xe_z\right)\right\rangle_{B_p^\sigma(\mathbb{B}_n)} &= \left\langle \mathbb{M}_\varphi f, xe_z\right\rangle_{B_p^\sigma(\mathbb{B}_n;\ell^2)} = \sum_{i=1}^\infty \left\langle \varphi_i f, x_i e_z\right\rangle_{B_p^\sigma(\mathbb{B}_n)} \\
&= \sum_{i=1}^\infty \overline{x_i}\varphi_i\left(z\right) f\left(z\right) = \sum_{i=1}^\infty \overline{x_i}\varphi_i\left(z\right)\left\langle f, e_z\right\rangle_{B_p^\sigma(\mathbb{B}_n)} \\
&= \sum_{i=1}^\infty \left\langle f, \overline{\varphi_i\left(z\right)} x_i e_z\right\rangle_{B_p^\sigma(\mathbb{B}_n)} = \left\langle f, \left\langle x, \varphi\left(z\right)\right\rangle_{\ell^2} e_z\right\rangle_{B_p^\sigma(\mathbb{B}_n)},
\end{aligned}$$

shows that

$$\mathbb{M}_\varphi^*\left(xe_z\right) = \left\langle x, \varphi\left(z\right)\right\rangle_{\ell^2} e_z.$$

This yields

$$\begin{aligned}
\left|\left\langle x, \varphi\left(z\right)\right\rangle_{\ell^2}\right| \left\|e_z\right\|_{B_p^\sigma(\mathbb{B}_n)^*} &= \left\|\mathbb{M}_\varphi^*\left(xe_z\right)\right\|_{B_p^\sigma(\mathbb{B}_n)^*} \\
&\le \left\|\mathbb{M}_\varphi^*\right\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)^*\to B_p^\sigma(\mathbb{B}_n)^*} \left\|xe_z\right\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)^*} \\
&= \left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \left|x\right| \left\|e_z\right\|_{B_p^\sigma(\mathbb{B}_n)^*},
\end{aligned}$$

which gives

$$\left|\varphi\left(z\right)\right| = \sup_{x\neq 0} \frac{\left|\left\langle x, \varphi\left(z\right)\right\rangle_{\ell^2}\right|}{\left|x\right|} \le \left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}, \qquad z \in \mathbb{B}_n,$$

and completes the proof of Lemma 8.

In order to deal with functions f on $\mathbb{B}_n$ that are not necessarily holomorphic, we use a notion of higher order derivative D^m introduced in [6] that is based on iterating D_a rather than $\widetilde{\nabla}$.

Definition 6. *For $m \in \mathbb{N}$ and $f \in C^\infty\left(\mathbb{B}_n;\ell^2\right)$ smooth in $\mathbb{B}_n$ we define $\Theta^m f\left(a,z\right) = D_a^m f\left(z\right)$ for $a, z \in \mathbb{B}_n$, and then set*

$$D^m f\left(z\right) = \Theta^m f\left(z,z\right) = D_z^m f\left(z\right), \qquad z \in \mathbb{B}_n.$$

Note that in this definition, we iterate the operator D_z holding z fixed, and then evaluate the result at the same z. If we combine Lemmas 6 and 7 we obtain that for $f \in H\left(\mathbb{B}_n;\ell^2\right)$,

$$\left\|f\right\|_{B_{p,m}^\sigma(\mathbb{B}_n;\ell^2)} \approx \sum_{j=0}^{m-1} \left|\nabla^j f\left(0\right)\right| + \left(\int_{\mathbb{B}_n} \left|\left(1-\left|z\right|^2\right)^\sigma D^m f\left(z\right)\right|^p d\lambda_n\left(z\right)\right)^{\frac{1}{p}}.$$

6.1. **Real variable analogues of Besov-Sobolev spaces.** In order to handle the operators arising from integration by parts formulas below, we will need yet more general equivalent norms on $B_{p,m}^\sigma\left(\mathbb{B}_n;\ell^2\right)$.

Definition 7. *We denote by $\mathcal{X}^m$ the vector of all differential operators of the form $X_1 X_2 ... X_m$ where each X_i is either $1-|z|^2$ times the identity operator I, the operator $\overline{D}$, or the operator $\left(1-|z|^2\right) R$. Just as in Definition 6, we calculate the products $X_1 X_2 ... X_m$ by composing $\overline{D_a}$ and $\left(1-|a|^2\right) R$ and then setting $a=z$ at the end. Note that $\overline{D_a}$ and $\left(1-|a|^2\right) R$ commute since the first is an antiholomorphic derivative and the coefficient z in $R = z \cdot \nabla$ is holomorphic. Similarly we denote by $\mathcal{Y}^m$ the corresponding products of $\left(1-|z|^2\right) I$, D (instead of $\overline{D}$) and $\left(1-|z|^2\right) R$.*

In the iterated derivative $\mathcal{X}^m$ we are differentiating only with the *antiholomorphic* derivative $\overline{D}$ or the *holomorphic* derivative R. When f is holomorphic, we thus have $\mathcal{X}^m f \sim \left\{\left(1-|z|^2\right)^m R^k f\right\}_{k=0}^m$. The reason we allow $1-|z|^2$ times the identity I to occur in $\mathcal{X}^m$ is that this produces a norm (as opposed to just a seminorm) without including the term $\sum_{k=0}^{m-1}\left|\nabla^k f(0)\right|$. We define the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ for *smooth* f on the ball $\mathbb{B}_n$ by

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \equiv \left(\sum_{k=0}^m \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} R^k f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}},$$

and note that provided $m+\sigma > \frac{n}{p}$, this provides an equivalent norm for the Besov-Sobolev space $B^\sigma_p\left(\mathbb{B}_n;\ell^2\right)$ of holomorphic functions on $\mathbb{B}_n$. These considerations motivate the following two definitions of a *real-variable* analogue of the norm $\|\cdot\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$.

Definition 8. *We define the* norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ *and* $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ *for* $f=(f_i)_{i=1}^\infty$ *smooth on the ball* $\mathbb{B}_n$ *by*

$$\begin{aligned} \|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^\sigma \mathcal{X}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}, \\ \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &\equiv \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^\sigma \mathcal{Y}^m f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \end{aligned} \tag{6.7}$$

It is *not* true that either of the norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ or $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are independent of m for large m when acting on smooth functions. However, Lemmas 6 and 7 show the equivalence of norms when restricted to holomorphic vector functions:

Lemma 9. *Let $1<p<\infty$, $\sigma \geq 0$ and $m > 2\left(\frac{n}{p}-\sigma\right)$. Then for f a holomorphic vector function we have*

$$\|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \approx \|f\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}. \tag{6.8}$$

The norms $\|\cdot\|_{\Lambda^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ arise in the integration by parts in iterated Charpentier kernels in Section 8, while the norms $\|\cdot\|_{\Phi^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ are useful for estimating the holomorphic function g in the Koszul complex. For this latter purpose we will use the following multilinear inequality whose scalar version is, after translating notation, Theorem 3.5 in [20]. The extension to ℓ^2-valued functions is routine but again, for the convenience of the reader, we give a detailed proof in the appendix.

Proposition 3. *Suppose that* $1 < p < \infty$, $0 \leq \sigma < \infty$, $M \geq 1$, $m > 2\left(\frac{n}{p} - \sigma\right)$ *and* $\alpha = (\alpha_0, ..., \alpha_M) \in \mathbb{Z}_+^{M+1}$ *with* $|\alpha| = m$. *For* $g \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ *and* $h \in B_p^\sigma(\mathbb{B}_n)$ *we have,*

$$\int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{p\sigma} |(\mathcal{Y}^{\alpha_1} g)(z)|^p \ldots |(\mathcal{Y}^{\alpha_M} g)(z)|^p |(\mathcal{Y}^{\alpha_0} h)(z)|^p \, d\lambda_n(z)$$
$$\leq C_{n,M,\sigma,p} \left(\|\mathbb{M}_g\|^{Mp}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \right) \|h\|^p_{B_p^\sigma(\mathbb{B}_n)}.$$

Remark 6. *The inequalities for* $M = 1$ *in Proposition 3 actually characterize multipliers* g *in the sense that a function* $g \in B_p^\sigma\left(\mathbb{B}_n;\ell^2\right) \cap H^\infty\left(\mathbb{B}_n;\ell^2\right)$ *is in* $M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$ if and only if *the inequalities with* $M = 1$ *in Proposition 3 hold. This follows from noting that each term in the Leibniz expansion of* $\mathcal{Y}^m(gh)$ *occurs on the left side of the display above with* $M = 1$.

6.1.1. *Three crucial inequalities.* In order to establish appropriate inequalities for the Charpentier solution operators, we will need to control terms of the form $(\overline{z-w})^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w)$, $\overline{D^m_{(z)}} \triangle(w,z)$, $\overline{D^m_{(z)}}\left\{(1 - w\overline{z})^k\right\}$and $R^m_{(z)}\left\{(1 - \overline{w}z)^k\right\}$ inside the integral for T as given in the integration by parts formula in Lemma 3 above. Here we are using the subscript (z) in parentheses to indicate the variable being differentiated. This is to avoid confusion with the notation D_a introduced in (6.1). We collect the necessary estimates in the following proposition.

Proposition 4. *For* $z, w \in \mathbb{B}_n$ *and* $m \in \mathbb{N}$, *we have the following three crucial estimates:*

(6.9)
$$\left|(\overline{z-w})^\alpha \frac{\partial^m}{\partial \overline{w}^\alpha} F(w)\right| \leq C\left(\frac{\sqrt{\triangle(w,z)}}{1 - |w|^2}\right)^m \left|\overline{D}^m F(w)\right|, \qquad F \in H\left(\mathbb{B}_n;\ell^2\right), m = |\alpha|.$$

(6.10)
$$\begin{aligned} \left|D_{(z)} \triangle(w,z)\right| &\leq C\left\{\left(1 - |z|^2\right) \triangle(w,z)^{\frac{1}{2}} + \triangle(w,z)\right\}, \\ \left|\left(1 - |z|^2\right) R_{(z)} \triangle(w,z)\right| &\leq C\left(1 - |z|^2\right)\sqrt{\triangle(w,z)}, \end{aligned}$$

(6.11)
$$\begin{aligned} \left|D^m_{(z)}\left\{(1 - \overline{w}z)^k\right\}\right| &\leq C\,|1 - \overline{w}z|^k \left(\frac{1 - |z|^2}{|1 - \overline{w}z|}\right)^{\frac{m}{2}}, \\ \left|\left(1 - |z|^2\right)^m R^m_{(z)}\left\{(1 - \overline{w}z)^k\right\}\right| &\leq C\,|1 - \overline{w}z|^k \left(\frac{1 - |z|^2}{|1 - \overline{w}z|}\right)^m. \end{aligned}$$

Proof: To prove (6.9) we view D_a as a differentiation operator in the variable w so that

$$D_a = -\nabla_w\left\{\left(1 - |a|^2\right) P_a + \sqrt{1 - |a|^2} Q_a\right\}.$$

A basic calculation is then:

$$
\begin{aligned}
&(1-\overline{a}z)\,\varphi_a(z)\cdot(D_a)^t \\
&\quad=\left\{P_a(z-a)+\sqrt{1-|a|^2}Q_a(z-a)\right\} \\
&\qquad\qquad\cdot\left\{\left(1-|a|^2\right)P_a\nabla_w+\sqrt{1-|a|^2}Q_a\nabla_w\right\} \\
&\quad=P_a(z-a)\left(1-|a|^2\right)P_a\nabla_w \\
&\qquad\qquad+\sqrt{1-|a|^2}Q_a(z-a)\sqrt{1-|a|^2}Q_a\nabla_w \\
&\quad=\left(1-|a|^2\right)(z-a)\cdot\nabla_w.
\end{aligned}
$$

From this we conclude the inequality

$$
\begin{aligned}
\left|(z_i-a_i)\frac{\partial}{\partial w_i}F(w)\right| &\leq |(z-a)\cdot\nabla F(w)| \\
&\leq \left|\frac{1-\overline{a}z}{1-|a|^2}\varphi_a(z)\right||D_aF(w)| \\
&= \frac{\sqrt{\triangle(a,z)}}{1-|a|^2}|D_aF(w)|,
\end{aligned}
$$

as well as its conjugate

$$
\left|\overline{(z_i-a_i)}\frac{\partial}{\partial\overline{w_i}}F(w)\right|\leq C\frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\left|\overline{D_a}F(w)\right|.
$$

Moreover, we can iterate this inequality to obtain

$$
\left|\left(\overline{z-a}\right)^\alpha\frac{\partial^m}{\partial\overline{w}^\alpha}F(w)\right|\leq C\left(\frac{\sqrt{\triangle(a,z)}}{1-|a|^2}\right)^m\left|\left(\overline{D_a}\right)^mF(w)\right|,
$$

for a multi-index of length m. With $a=w$ this becomes the first estimate (6.9).

To see the second estimate (6.10), recall from (6.1) that

$$
D_af(z)=-\left\{\left(1-|a|^2\right)P_a\nabla f(z)+\left(1-|a|^2\right)^{\frac{1}{2}}Q_a\nabla f(z)\right\}.
$$

We let $a=z$. By the unitary invariance of

$$
\triangle(w,z)=|1-\overline{w}z|^2-\left(1-|z|^2\right)\left(1-|w|^2\right),
$$

we may assume that $z=(|z|,0,...,0)$. Then we have

$$
\begin{aligned}
\frac{\partial}{\partial z_j}\triangle(w,z) &= \frac{\partial}{\partial z_j}\left\{(1-\overline{w}z)(1-\overline{z}w)-(1-\overline{z}z)\left(1-|w|^2\right)\right\} \\
&= -\overline{w_j}(1-\overline{z}w)+\overline{z_j}\left(1-|w|^2\right) \\
&= (\overline{z_j}-\overline{w_j})+\overline{w_j}(\overline{z}w)-\overline{z_j}|w|^2 \\
&= (\overline{z_j}-\overline{w_j})\left(1-|z|^2\right)+\overline{z_j}|z|^2-\overline{w_j}|z|^2+\overline{w_j}(\overline{z}w)-\overline{z_j}|w|^2 \\
&= (\overline{z_j}-\overline{w_j})\left(1-|z|^2\right)+\overline{z_j}\left(|z|^2-|w|^2\right)+\overline{w_j}(\overline{z}(w-z)).
\end{aligned}
$$

Now $Q_z\nabla f = \left(0, \frac{\partial f}{\partial z_2}, ..., \frac{\partial f}{\partial z_n}\right)$ and thus a typical term in $Q_z\nabla\triangle$ is $\frac{\partial}{\partial z_j}\triangle(w,z)$ with $j\geq 2$. From $z=(|z|,0,...,0)$ and $j\geq 2$ we have $z_j=0$ and so

$$\frac{\partial}{\partial z_j}\triangle(w,z) = (\overline{z_j}-\overline{w_j})\left(1-|z|^2\right) - (\overline{z_j}-\overline{w_j})\left(\overline{z}(w-z)\right), \qquad j\geq 2.$$

Now (3.4) implies

$$\triangle(w,z) = \left(1-|z|^2\right)|w-z|^2 + |\overline{z}(w-z)|^2, \tag{6.12}$$

which together with the above shows that

$$\begin{aligned}\sqrt{1-|z|^2}\,|Q_z\nabla\triangle(w,z)| &\leq C|z-w|\left(1-|z|^2\right)^{\frac{3}{2}} \\ &\quad +C\sqrt{1-|z|^2}\,|z-w|\,|\overline{z}(w-z)| \\ &\leq C\left(1-|z|^2\right)\triangle(w,z)^{\frac{1}{2}} + C\triangle(w,z).\end{aligned}\tag{6.13}$$

As for $P_z\nabla D = \left(\frac{\partial f}{\partial z_1}, 0, ..., 0\right)$ we use (6.12) to obtain

$$\begin{aligned}|P_z\nabla\triangle(w,z)| &= \left|(\overline{z_1}-\overline{w_1})\left(1-|z|^2\right) + \overline{z_1}\left(|z|^2-|w|^2\right) + \overline{w_1}\overline{z}(w-z)\right| \\ &\leq |z-w|\left(1-|z|^2\right) + \left||z|^2-|w|^2\right| + |\overline{z}(w-z)| \\ &\leq C\sqrt{\triangle(w,z)} + 2\,||z|-|w||.\end{aligned}$$

However,

$$\begin{aligned}\triangle(w,z) &\geq (1-|w|\,|z|)^2 - \left(1-|z|^2\right)\left(1-|w|^2\right) \\ &= 1-2|w|\,|z| + |w|^2|z|^2 - \left\{1-|z|^2-|w|^2+|z|^2|w|^2\right\} \\ &= |z|^2+|w|^2-2|w|\,|z| = (|z|-|w|)^2\end{aligned}$$

and so altogether we have the estimate

$$|P_z\nabla\triangle(w,z)| \leq C\sqrt{\triangle(w,z)}. \tag{6.14}$$

Combining (6.13) and (6.14) with the definition (6.1) completes the proof of the first line in (6.10). The second line in (6.10) follows from (6.14) since $R_{(z)} = P_z\nabla$.

To prove the third estimate (6.11) we compute:

$$\begin{aligned}D_{(z)}(1-\overline{w}z)^k &= k(1-\overline{w}z)^{k-1}D_{(z)}(1-\overline{w}z) \\ &= k(1-\overline{w}z)^{k-1}\left\{\left(1-|z|^2\right)P_z\nabla + \sqrt{1-|z|^2}Q_z\nabla\right\}(1-\overline{w}z) \\ &= -k(1-\overline{w}z)^{k-1}\left\{\left(1-|z|^2\right)P_z\overline{w} + \sqrt{1-|z|^2}Q_z\overline{w}\right\}; \\ R_{(z)}(1-\overline{w}z)^k &= k(1-\overline{w}z)^{k-1}(-\overline{w}z).\end{aligned}$$

Since $|w|^2+|a|^2\leq 2$ we have

$$\begin{aligned}|Q_z\overline{w}|^2 &= |Q_z(\overline{w}-\overline{z})|^2 \leq |\overline{w}-\overline{z}|^2, \\ &= |w|^2+|z|^2-2\operatorname{Re}(w\overline{z}) \\ &\leq 2\operatorname{Re}(1-w\overline{z}) \leq 2|1-w\overline{z}|,\end{aligned}$$

which yields

$$\begin{aligned}\left|D_{(z)}\left\{(1-\overline{w}z)^k\right\}\right| &\leq C\left|1-\overline{w}z\right|^k\left\{\frac{\left(1-|z|^2\right)+\sqrt{\left(1-|z|^2\right)\left|1-\overline{w}z\right|}}{\left|1-\overline{w}z\right|}\right\}\\ &\leq C\left|1-\overline{w}z\right|^k\sqrt{\frac{1-|z|^2}{\left|1-\overline{w}z\right|}}.\end{aligned}$$

Iteration then yields (6.11).

7. Schur's Test

Here we characterize boundedness of the positive operators that arise as majorants of the solution operators below. The case $c=0$ of the following lemma is Theorem 2.10 in [36].

Lemma 10. *Let $a,b,c,t\in\mathbb{R}$. Then the operator*

$$T_{a,b,c}f(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^b\left(\sqrt{\triangle(w,z)}\right)^c}{\left|1-w\overline{z}\right|^{n+1+a+b+c}}f(w)\,dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n;\left(1-|w|^2\right)^t dV(w)\right)$ if and only if $c>-2n$ and

$$-pa<t+1<p(b+1). \tag{7.1}$$

We sketch the proof for the case $c\neq 0$ when $p=2$ and $t=-n-1$. Let $\psi_\varepsilon(\zeta)=\left(1-|\zeta|^2\right)^\varepsilon$ and recall that $\sqrt{\triangle(w,z)}=\left|1-w\overline{z}\right|\left|\varphi_z(w)\right|$. We compute conditions on a, b, c and ε such that we have

$$T_{a,b,c}\psi_\varepsilon(z)\leq C\psi_\varepsilon(z) \text{ and } T^*_{a,b,c}\psi_\varepsilon(w)\leq C\psi_\varepsilon(w), \qquad z,w\in\mathbb{B}_n,$$

where $T^*_{a,b,c}$ denotes the dual relative to $L^2(\lambda_n)$. For this we take $\varepsilon\in\mathbb{R}$ and compute

$$T_{a,b,c}\psi_\varepsilon(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{n+1+b+\varepsilon}\left|\varphi_z(w)\right|^c}{\left|1-w\overline{z}\right|^{n+1+a+b}}d\lambda_n(w).$$

Note that the integral defining $T_{a,b,c}\psi_\varepsilon(z)$ is finite if and only if $\varepsilon>-b-1$. Now in this integral make the change of variable $w=\varphi_z(\zeta)$ and use that λ_n is invariant to obtain

$$T_{a,b,c}\psi_\varepsilon(z)=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^a\left(1-\left|\varphi_z(\zeta)\right|^2\right)^{n+1+b+\varepsilon}|\zeta|^c}{\left|1-\overline{\varphi_z(\zeta)}z\right|^{n+1+a+b}(1-|\zeta|^2)^{n+1}}dV(\zeta).$$

Plugging the identities

$$\begin{aligned}1-\varphi_z(\zeta)\overline{z} &= 1-\langle\varphi_z(\zeta),\varphi_z(0)\rangle=\frac{1-|z|^2}{1-\zeta\overline{z}},\\ 1-\left|\varphi_z(\zeta)\right|^2 &= 1-\langle\varphi_z(\zeta),\varphi_z(\zeta)\rangle=\frac{\left(1-|z|^2\right)\left(1-|\zeta|^2\right)}{\left|1-\zeta\overline{z}\right|^2},\end{aligned}$$

into the formula for $T_{a,b,c}\psi_\varepsilon(z)$ we obtain

$$T_{a,b,c}\psi_\varepsilon(z) = \psi_\varepsilon(z)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{b+\varepsilon}|\zeta|^c}{|1-\zeta\overline{z}|^{n+1+b-a+2\varepsilon}}dV(\zeta).$$

Then from Theorem 1.12 in [36] we obtain that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{\alpha}}{|1-\zeta\overline{z}|^{\beta}}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. Provided $c>-2n$ it is now easy to see that we also have

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{\alpha}|\zeta|^c}{|1-\zeta\overline{z}|^{\beta}}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. It now follows from the above that

$$T_{a,b,c}\psi_\varepsilon(z)\le C\psi_\varepsilon(z),\qquad z\in\mathbb{B}_n,$$

if and only if

$$-b-1<\varepsilon<a.$$

Arguing as above and provided $c>-2n$, we obtain

$$T^*_{a,b,c}\psi_\varepsilon(w)\le C\psi_\varepsilon(w),\qquad w\in\mathbb{B}_n,$$

if and only if

$$-a+n<\varepsilon<b+n+1.$$

Altogether then there is $\varepsilon\in\mathbb{R}$ such that $h=\sqrt{\psi_\varepsilon}$ is a Schur function for $T_{a,b,c}$ on $L^2(\lambda_n)$ in Schur's Test (as given in Theorem 2.9 on page 51 of [36]) if and only if

$$\max\{-a+n,-b-1\}<\min\{a,b+n+1\}.$$

This is equivalent to $-2a<-n<2(b+1)$, which is (7.1) in the case $p=2, t=-n-1$. This completes the proof (in this case) that (7.1) implies the boundedness of $T_{a,b,c}$ on $L^2(\lambda_n)$. The converse is easy - see for example the argument for the case $c=0$ on page 52 of [36].

See the appendix for a more detailed proof of Lemma 10.

Remark 7. *We will also use the trivial consequence of Lemma 10 that the operator*

$$T_{a,b,c,d}f(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^b\left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c+d}}f(w)\,dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n;\left(1-|w|^2\right)^t dV(w)\right)$ *if* $c>-2n$, $d\le 0$ *and (7.1) holds. This is simply because* $|1-w\overline{z}|\le 2$.

8. Operator estimates

We must show that $f = \Omega_0^1 h - \Lambda_g \Gamma_0^2 \in B_p^\sigma\left(\mathbb{B}_n;\ell^2\right)$ where Γ_0^2 is an antisymmetric 2-tensor of $(0,0)$-forms that solves

$$\overline{\partial}\Gamma_0^2 = \Omega_1^2 h - \Lambda_g \Gamma_1^3,$$

and inductively where Γ_q^{q+2} is an alternating $(q+2)$-tensor of $(0,q)$-forms that solves

$$\overline{\partial}\Gamma_q^{q+2} = \Omega_{q+1}^{q+2} h - \Lambda_g \Gamma_{q+1}^{q+3},$$

up to $q = n-1$ (since $\Gamma_n^{n+2} = 0$ and the $(0,n)$-form Ω_n^{n+1} is $\overline{\partial}$-closed). Using the Charpentier solution operators $\mathcal{C}_{n,s}^{0,q}$ on $(0,q+1)$-forms we then get

$$\begin{aligned}
f &= \Omega_0^1 h - \Lambda_g \Gamma_0^2 \\
&= \Omega_0^1 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_g \Gamma_1^3\right) \\
&= \Omega_0^1 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\left(\Omega_1^2 h - \Lambda_g \mathcal{C}_{n,s_2}^{0,1}\left(\Omega_2^3 h - \Lambda_g \Gamma_2^4\right)\right) \\
&\ \ \vdots \\
&= \Omega_0^1 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\Omega_1^2 h + \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\Lambda_g \mathcal{C}_{n,s_2}^{0,1}\Omega_2^3 h - \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\Lambda_g \mathcal{C}_{n,s_2}^{0,1}\Lambda_g \mathcal{C}_{n,s_3}^{0,2}\Omega_3^4 h - \ldots \\
&\quad + (-1)^n \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\ldots\Lambda_g \mathcal{C}_{n,s_n}^{0,n-1}\Omega_n^{n+1} h \\
&\equiv \mathcal{F}^0 + \mathcal{F}^1 + \ldots + \mathcal{F}^n.
\end{aligned}$$

The goal is to establish

$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)} \leq C_{n,\sigma,p,\delta}(g)\,\|h\|_{B_p^\sigma(\mathbb{B}_n)},$$

which we accomplish by showing that

$$\|\mathcal{F}^\mu\|_{B_{p,m_1}^\sigma(\mathbb{B}_n;\ell^2)} \leq C_{n,\sigma,p,\delta}(g)\,\|h\|_{\Lambda_{p,m_\mu}^\sigma(\mathbb{B}_n)}, \qquad 0 \leq \mu \leq n, \tag{8.1}$$

for a choice of integers m_μ satisfying

$$\frac{n}{p} - \sigma < m_1 < m_2 < \ldots < m_\ell < \ldots < m_n.$$

Recall that we defined both of the norms $\|F\|_{B_{p,m_\mu}^\sigma(\mathbb{B}_n;\ell^2)}$ and $\|F\|_{\Lambda_{p,m_\mu}^\sigma(\mathbb{B}_n;\ell^2)}$ for smooth vector functions F in the ball $\mathbb{B}_n$.

> **Note on constants:** We often indicate via subscripts, such as n,σ,p,δ, the important parameters on which a given constant C depends, especially when the constant appears in a basic inequality. However, at times in mid-argument, we will often revert to suppressing some or all of the subscripts in the interests of readability.

The norms $\|\cdot\|_{\Lambda_{p,m}^\sigma(\mathbb{B}_n;\ell^2)}$ in (6.7) above will now be used to estimate the composition of Charpentier solution operators in each function

$$\mathcal{F}^\mu = \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\ldots\Lambda_g \mathcal{C}_{n,s_\mu}^{0,\mu-1}\Omega_\mu^{\mu+1} h$$

as follows. More precisely we will use the specialized variants of the seminorms given by

$$\|F\|_{\Lambda_{p,m',m''}^\sigma(\mathbb{B}_n;\ell^2)}^p \equiv \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma \left\{\left(1-|z|^2\right)^{m'} R^{m'}\right\}\overline{D}^{m''} F(z)\right|^p d\lambda_n(z),$$

where we take m'' derivatives in $\overline{D}$ followed by m' derivatives in the invariant radial operator $\left(1-|z|^2\right)R$. Recall from Definition 7 that $\mathcal{X}^m$ denotes the vector of all differential operators of the form $X_1X_2...X_m$ where each X_i is either I, $\overline{D}$, or $\left(1-|z|^2\right)R$, and where by definition $1-|z|^2$ is held constant in composing operators. It will also be convenient at times to use the notation

$$\mathcal{R}^m \equiv \left(1-|z|^2\right)^m \left(R^k\right)_{k=0}^m, \tag{8.2}$$

which should cause no confusion with the related operators $\mathcal{R}_b^m$ in (4.8) introduced in the remark following Corollary 3. Note that $\mathcal{R}^m$ is simply $\mathcal{X}^m$ when none of the operators $\overline{D}$ appear. We will make extensive use the multilinear estimate in Proposition 3.

Let us fix our attention on the function $\mathcal{F}^\mu = \mathcal{F}_0^\mu$ and write

$$\begin{aligned}
\mathcal{F}_0^\mu &= \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\left\{\Lambda_g \mathcal{C}_{n,s_2}^{0,1}...\Lambda_g \mathcal{C}_{n,s_\mu}^{0,\mu-1}\Omega_\mu^{\mu+1}h\right\} = \Lambda_g \mathcal{C}_{n,s_1}^{0,0}\left\{\mathcal{F}_1^\mu\right\}, \\
\mathcal{F}_1^\mu &= \Lambda_g \mathcal{C}_{n,s_2}^{0,1}\left\{\Lambda_g \mathcal{C}_{n,s_3}^{0,2}...\Lambda_g \mathcal{C}_{n,s_\mu}^{0,\mu-1}\Omega_\mu^{\mu+1}h\right\} = \Lambda_g \mathcal{C}_{n,s_2}^{0,1}\left\{\mathcal{F}_2^\mu\right\}, \\
\mathcal{F}_q^\mu &= \Lambda_g \mathcal{C}_{n,s_{q+1}}^{0,q}\left\{\mathcal{F}_{q+1}^\mu\right\}, \qquad etc,
\end{aligned}$$

where $\mathcal{F}_q^\mu$ is a $(0,q)$-form. We now perform the integration by parts in Lemma 5 in each iterated Charpentier operator $\mathcal{F}_q^\mu = \Lambda_g \mathcal{C}_{n,s_{q+1}}^{0,q}\left\{\mathcal{F}_{q+1}^\mu\right\}$ to obtain

$$\begin{aligned}
\mathcal{F}_q^\mu &= \Lambda_g \mathcal{C}_{n,s_{q+1}}^{0,q}\mathcal{F}_{q+1}^\mu \\
&= \sum_{j=0}^{m'_{q+1}-1} c'_{j,n,s_{q+1}}\Lambda_g \mathcal{S}_{n,s_{q+1}}\left(\overline{\mathcal{D}}^j\mathcal{F}_{q+1}^\mu\right)(z) \\
&\quad + \sum_{\ell=0}^{\mu} c_{\ell,n,s_{q+1}}\Lambda_g \Phi_{n,s_{q+1}}^\ell\left(\overline{\mathcal{D}}^{m'_{q+1}}\mathcal{F}_{q+1}^\mu\right)(z).
\end{aligned} \tag{8.3}$$

Now we compose these formulas for $\mathcal{F}_k^\mu$ to obtain an expression for $\mathcal{F}^\mu$ that is a complicated sum of compositions of the individual operators in (8.3) above. For now we will concentrate on the main terms $\Lambda_g \Phi_{n,s_{k+1}}^\mu\left(\overline{\mathcal{D}}^{m'_{k+1}}\mathcal{F}_{k+1}^\mu\right)$ that arise in the second sum above when $\ell=\mu$. We will see that the same considerations apply to any of the other terms in (8.3). Recall from Lemma 5 that the "boundary" operators $\mathcal{S}_{n,s_{q+1}}$ are projections of operators on $\partial\mathbb{B}_{s_q}$ to the ball $\mathbb{B}_n$ and have (balanced) kernels even simpler than those of the operators $\Phi_{n,s_{q+1}}^\ell$. The composition of these main terms is

$$\begin{aligned}
&\left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathcal{D}}^{m'_1}\right)\mathcal{F}_1^\mu \\
&= \left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathcal{D}}^{m'_1}\right)\left(\Lambda_g \Phi_{n,s_2}^\mu \overline{\mathcal{D}}^{m'_2}\right)\mathcal{F}_2^\mu \\
&= \left(\Lambda_g \Phi_{n,s_1}^\mu \overline{\mathcal{D}}^{m'_1}\right)\left(\Lambda_g \Phi_{n,s_2}^\mu \overline{\mathcal{D}}^{m'_2}\right)...\left(\Lambda_g \Phi_{n,s_\mu}^\mu \overline{\mathcal{D}}^{m'_\mu}\right)\Omega_\mu^{\mu+1}h.
\end{aligned} \tag{8.4}$$

At this point we would like to take absolute values inside all of these integrals and use the crucial inequalities in Proposition 4 to obtain a composition of positive operators of the type considered in Lemma 10. However, there is a difficulty in using the crucial inequality (6.9) to estimate the derivative $\overline{\mathcal{D}}^m$ on $(0,q+1)$-forms

η given by (4.6):

$$\overline{\mathcal{D}}^{m}\eta(z)=\sum_{|J|=q}\sum_{k\notin J}\sum_{|\alpha|=m}(-1)^{\mu(k,J)}\overline{(w_k-z_k)}\overline{(w-z)}^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}\eta_{J\cup\{k\}}(w).$$

The problem is that the factor $\overline{(w_k-z_k)}$ has no derivative $\frac{\partial}{\partial\overline{w_k}}$ naturally associated with it, as do the other factors in $\overline{(w-z)}^{\alpha}$. We refer to the factor $\overline{(w_k-z_k)}$ as a *rogue* factor, as it requires special treatment in order to apply (6.9). Note that we cannot simply estimate $\overline{(w_k-z_k)}$ by $|w-z|$ because this is much larger in general than the estimate $\sqrt{\triangle(w,z)}$ obtained in (6.9) (where the difference in size between $|w-z|$ and $\sqrt{\triangle(w,z)}$ is compensated by the difference in size between $\frac{\partial}{\partial\overline{w_k}}$ and $\overline{D}$).

We now describe how to circumvent this difficulty in the composition of operators in (8.4). Let us write each $\overline{\mathcal{D}}^{m'_{q+1}}\mathcal{F}^{\mu}_{q+1}$ as

$$\sum_{|J|=q}\sum_{k\notin J}\sum_{|\alpha|=m'_{q+1}}(-1)^{\mu(k,J)}\overline{(w_k-z_k)}\overline{(w-z)}^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}\left(\mathcal{F}^{\mu}_{q+1}\right)_{J\cup\{k\}}(w),$$

where $\left(\mathcal{F}^{\mu}_{q+1}\right)_{J\cup\{k\}}$ is the coefficient of the form $\mathcal{F}^{\mu}_{q+1}$ with differential $d\overline{w}^{J\cup\{k\}}$. We now replace each of these sums with just one of the summands, say

$$\overline{(w_k-z_k)}\overline{(w-z)}^{\alpha}\frac{\partial^m}{\partial\overline{w}^{\alpha}}\left(\mathcal{F}^{\mu}_{q+1}\right)_{J\cup\{k\}}(w). \tag{8.5}$$

Here the factor $\overline{(w_k-z_k)}$ is a *rogue* factor, not associated with a corresponding derivative $\frac{\partial}{\partial\overline{w_k}}$. We will refer to k as the *rogue* index associated with the *rogue* factor when it is not convenient to explicitly display the variables.

The key fact in treating the *rogue* factor $\overline{(w_k-z_k)}$ is that its presence in (8.5) means that the coefficient $\left(\mathcal{F}^{\mu}_{q+1}\right)_I$ of the form $\mathcal{F}^{\mu}_{q+1}$ that multiplies it *must* have k in the multi-index I. Since $\mathcal{F}^{\mu}_{q+1}=\Lambda_g\mathcal{C}^{0,q+1}_{n,s_{q+2}}\left\{\mathcal{F}^{\mu}_{q+2}\right\}$, the form of the ameliorated Charpentier kernel $\mathcal{C}^{0,q+1}_{n,s_{q+2}}$ in Theorem 6 shows that the coefficients of $\mathcal{C}^{0,q+1}_{n,s_{q+2}}(w,z)$ that multiply the *rogue* factor *must* have the differential $d\overline{z_k}$ in them. In turn, this means that the differential $d\overline{w_k}$ must be *missing* in the coefficient of $\mathcal{C}^{0,q+1}_{n,s_{q+2}}(w,z)$, and hence finally that the coefficients $\left(\mathcal{F}^{\mu}_{q+2}\right)_H$ with multi-index H that survive the wedge products in the integration *must* have $k\in H$. This observation can be repeated, and we now derive an important consequence.

Returning to (8.4), each summand in $\overline{\mathcal{D}}^{m'_{q+1}}\mathcal{F}^{\mu}_{q+1}$ has a *rogue* factor with associated *rogue* index k_{q+1}. Thus the function in (8.4) is a sum of terms of the form

$$\begin{aligned}\left(\Lambda_g\Phi^{\mu}_{n,s_1}\overline{(w_{k_1}-z_{k_1})}\overline{\mathcal{Z}}^{m'_1}\right)\circ\left(\Lambda_g\Phi^{\mu}_{n,s_2}\overline{(w_{k_2}-z_{k_2})}\overline{\mathcal{Z}}^{m'_2}\right)_{I_1}\circ\\ \dots\circ\left(\Lambda_g\Phi^{\nu}_{n,s_\nu}\overline{(w_{k_\nu}-z_{k_\nu})}\overline{\mathcal{Z}}^{m'_\nu}\right)_{I_{\nu-1}}\circ\\ \dots\circ\left(\Lambda_g\Phi^{\mu-1}_{n,s_\mu}\overline{(w_{k_\mu}-z_{k_\mu})}\overline{\mathcal{Z}}^{m'_\mu}\right)_{I_{\mu-1}}\circ\left(\Omega^{\mu+1}_{\mu}h\right)_{I_\mu},\end{aligned}$$

where the subscript I_ν on the form $\Lambda_g\Phi^{\nu}_{n,s_\nu}\overline{(w_{k_\nu}-z_{k_\nu})}\overline{\mathcal{Z}}^{m'_\nu}$ indicates that we are composing with the component of $\Lambda_g\Phi^{\nu}_{n,s_\nu}\overline{(w_{k_\nu}-z_{k_\nu})}\overline{\mathcal{Z}}^{m'_\nu}$ corresponding to the

multi-index $I_{\nu-1}$, i.e. the component with the differential $d\overline{z}^{I_{\nu-1}}$. The notation will become exceedingly unwieldy if we attempt to identify the different variables associated with each of the iterated integrals, so we refrain from this in general. The considerations of the previous paragraph now show that we must have $\{k_1\} = I_1$, $\{k_2\} \cup I_1 = I_2$ and more generally

$$\{k_\nu\} \cup I_{\nu-1} = I_\nu, \qquad 1 < \nu \le \mu.$$

In particular we see that the associated *rogue* indices $k_1, k_2, ...k_\mu$ are all distinct and that as sets

$$\{k_1, k_2, ..., k_\mu\} = I_\mu.$$

If we denote by ζ the variable in the final form $\Omega_\mu^{\mu+1}h$, we can thus write each *rogue* factor $\overline{(w_{k_\nu} - z_{k_\nu})}$ as

$$\overline{(w_{k_\nu} - z_{k_\nu})} = \overline{(w_{k_\nu} - \zeta_{k_\nu})} - \overline{(z_{k_\nu} - \zeta_{k_\nu})},$$

and since $k_\nu \in I_\mu$, there is a factor of the form $\frac{\partial}{\partial\overline{\zeta_{k_\nu}}} \frac{\partial^{|\beta|} g_i}{\partial\overline{\zeta}^\beta}$ in each summand of the component $\left(\Omega_\mu^{\mu+1}h\right)_{I_\mu}$ of $\Omega_\mu^{\mu+1}h$. So we are able to associate the *rogue* factor $\overline{(w_{k_\nu} - z_{k_\nu})}$ with derivatives of g as follows:

$$\left\{\overline{(w_{k_\nu} - \zeta_{k_\nu})} \frac{\partial}{\partial\overline{\zeta_{k_\nu}}}\right\} \frac{\partial^{|\beta|} g_i}{\partial\overline{\zeta}^\beta} - \left\{\overline{(z_{k_\nu} - \zeta_{k_\nu})} \frac{\partial}{\partial\overline{\zeta_{k_\nu}}}\right\} \frac{\partial^{|\gamma|} g_j}{\partial\overline{\zeta}^\gamma}. \tag{8.6}$$

Thus it is indeed possible to

(1) apply the radial integration by parts in Corollary 3,
(2) then take absolute values and ℓ^2-norms inside all the integrals,
(3) and then apply the crucial inequalities in Proposition 4.

One of the difficulties remaining after this is that we are now left with additional factors of the form

$$\frac{\sqrt{\triangle(w,\zeta)}}{1-|w|^2}, \frac{\sqrt{\triangle(z,\zeta)}}{1-|z|^2}$$

that result from an application of (6.9) to the derivatives in (8.6). These factors are still *rogue* in the sense that the variable pairs occurring in them, namely (w,ζ) and (z,ζ), do not consist of consecutive variables in the iterated integrals of (8.4). This is rectified by using the fact that $d(w,z) = \sqrt{\triangle(w,z)}$ is a quasimetric, which in turn follows from the identity

$$\sqrt{\triangle(w,z)} = |1 - w\overline{z}| \, |\varphi_z(w)| = \delta(w,z)^2 \rho(w,z),$$

where $\rho(w,z) = |\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [36]) and where $\delta(w,z) = |1 - w\overline{z}|^{\frac{1}{2}}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [24]). Using the quasi-subadditivity of $d(w,z)$ we can, with some care, redistribute appropriate factors back to the iterated integrals where they can be favourably estimated using Lemma 10. It is simplest to illustrate this procedure in specific cases, so we defer further discussion of this point until we treat in detail the cases $\mu = 0, 1, 2$ below. We again emphasize that all of the above observations regarding *rogue* factors in (8.4) apply equally well to the *rogue* factors in the other terms $\Phi^\ell_{n,s_{q+1}}\left(\overline{\mathcal{D}}^{m'_q}\mathcal{F}^\mu_{q+1}\right)(z)$ in (8.3), as well as to the boundary terms $\mathcal{S}_{n,s_{q+1}}\left(\overline{\mathcal{D}}^j\mathcal{F}^\mu_{q+1}\right)(z)$ in (8.3).

The other difficulty remaining is that in order to obtain a favourable estimate using Lemma 10 for the iterated integrals resulting from the bullet items above, it is necessary to generate additional powers of $\left(1-|z|^2\right)$ (we are using z as a generic variable in the iterated integrals here). This is accomplished by applying the radial integrations by parts in Corollary 3 to the *previous* iterated integral. Of course such a possibility is impossible for the first of the iterated integrals, but there we are only applying the radial derivative R thanks to the fact that our candidate f from the Koszul complex is holomorphic. As a result, we see from (6.10) that $\left(1-|z|^2\right)R$, unlike D, generates positive powers of $1-|z|^2$ even when acting on $\triangle(w,z)$. This procedure is also best illustrated in specific cases and will be treated in the next subsection.

So ignoring these technical issues for the moment, the integrals that result from taking absolute values and ℓ^2-norms inside (8.4) are now estimated using Lemma 10 and Remark 7. Note that we only use *scalar-valued* Schur estimates since all the integrals to which Lemma 10 and Remark 7 are applied have positive integrands. Here is the rough idea. Suppose that $\{T_1,T_2,...,T_\mu\}$ is a collection of Charpentier solution operators and that for a sequence of large integers $\left\{m_1',m_1'',m_2',,m_2''...,m_{\mu+1}',m_{\mu+1}''\right\}$, we have the inequalities

$$\|T_jF\|_{\Lambda^\sigma_{p,m_j',m_j''}(\mathbb{B}_n;\ell^2)}\leq C_j\|F\|_{\Lambda^\sigma_{p,m_{j+1}',m_{j+1}''}(\mathbb{B}_n;\ell^2)},\qquad 1\leq j\leq \ell+1,\tag{8.7}$$

for the class of smooth functions F that arise as TG for some Charpentier solution operator T and some smooth G. Then we can estimate $\|T_1\circ T_2\circ...\circ T_\mu\Omega\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}$ by

$$\begin{aligned}
&\|T_1\circ T_2\circ...\circ T_\ell\Omega\|_{\Lambda^\sigma_{p,m_1',m_1''}(\mathbb{B}_n;\ell^2)}\\
\leq\ & C_1\|T_2\circ...\circ T_\ell\Omega\|_{\Lambda^\sigma_{p,m_2',m_2''}(\mathbb{B}_n;\ell^2)}\\
\leq\ & C_1C_2\|T_3\circ...\circ T_\ell\Omega\|_{\Lambda^\sigma_{p,m_3',m_3''}(\mathbb{B}_n;\ell^2)}\\
\leq\ & C_1C_2...C_\ell\|\Omega\|_{\Lambda^\sigma_{p,m_{\ell+1}',m_{\ell+1}''}(\mathbb{B}_n;\ell^2)}.
\end{aligned}$$

Finally we will show that if Ω is one of the forms Ω_q^{q+1} in the Koszul complex, then

$$\|\Omega\|_{\Lambda^\sigma_{p,m_{\ell+1}',m_{\ell+1}''}(\mathbb{B}_n;\ell^2)}\leq\|\Omega\|_{\Lambda^\sigma_{p,m_{\ell+1}'+m_{\ell+1}''}(\mathbb{B}_n;\ell^2)}\leq C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)},$$

and so altogether this proves that

$$\|f\|_{B^\sigma_p(\mathbb{B}_n;\ell^2)}\leq C_{n,\sigma,p,\delta}(g)\|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)}.$$

We now make some brief comments on how to obtain the inequalities in (8.7). Complete details will be given in the cases $\mu=0,1,2$ below, and the general case $0\leq\mu\leq n$ is no different than these three cases. We note that from (3.9) the kernel of $\mathcal{C}_n^{0,q}$ typically looks like a sum of terms

$$\frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}\left(\overline{z_j}-\overline{w_j}\right)\tag{8.8}$$

times a wedge product of differentials in which the differential $d\overline{w_j}$ is missing. We again emphasize that the *rogue* factor $\left(\overline{z_j}-\overline{w_j}\right)$ cannot simply be estimated by

$|\overline{z_j}-\overline{w_j}|$ as the formula (3.4) shows that

$$\sqrt{\triangle(w,z)}=\left|P_z(z-w)+\sqrt{1-|z|^2}Q_z(z-w)\right|$$

can be much smaller than $|z-w|$. As we mentioned above, it is possible to exploit the fact that any surviving term in the form $\Omega_\mu^{\mu+1}$ must then involve the derivative $\frac{\partial}{\partial\overline{w_j}}$ hitting a component of g. This permits us to absorb part of the complex tangential component of $z-w$ into the almost invariant derivative D which is larger than the usual gradient in the complex tangential directions. This results in a good estimate for the *rogue* factor $(\overline{z_j}-\overline{w_j})$ in (8.8) based on the smaller quantity $\sqrt{\triangle(w,z)}$. We have already integrated by parts to write (8.8) as (recall that the factors $\overline{z_j}-\overline{w_j}$ are already incorporated into $\overline{\mathcal{D}_z^m}\eta(w)$)

$$\int_{\mathbb{B}_n}\frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}\overline{\mathcal{D}^m}\eta(w)\,dV(w),$$

plus boundary terms which we ignore for the moment. Then we use the three crucial inequalities (6.9), (6.10) and (6.11);

$$\begin{aligned}
\left|(\overline{z_j}-\overline{w_j})\,\overline{\mathcal{D}_{z,w}^m}\Omega_\ell^{\ell+1}(w)\right| &\le \left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m+1}\left|\overline{D^m}\widehat{\Omega_\ell^{\ell+1}}(w)\right|,\\
\left|D_{(z)}\triangle(w,z)\right| &\le C\left(1-|z|^2\right)\triangle(w,z)^{\frac12}+\triangle(w,z),\\
\left|\left(1-|z|^2\right)R_{(z)}\triangle(w,z)\right| &\le C\left(1-|z|^2\right)\triangle(w,z)^{\frac12},\\
\left|D_{(z)}^m\left\{(1-\overline{w}z)^k\right\}\right| &\le C\,|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{\frac{m}{2}}\\
\left|\left(1-|z|^2\right)^m R_{(z)}^m\left\{(1-\overline{w}z)^k\right\}\right| &\le C\,|1-\overline{w}z|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^m,
\end{aligned}$$

to help show that the resulting iterated kernels can be factored (after accounting for all *rogue* factors $\overline{z_j}-\overline{w_j}$) into operators that satisfy the hypotheses of Lemma 10 or Remark 7 above.

Definition 9. *The expression $\widehat{\Omega_\ell^{\ell+1}}$ denotes the form $\Omega_\ell^{\ell+1}$ but with every occurrence of the derivative $\frac{\partial}{\partial\overline{w_j}}$ replaced by the derivative $\overline{D_j}$.*

Recall that each summand of $\Omega_\ell^{\ell+1}$ includes a product of exactly ℓ distinct derivatives $\frac{\partial}{\partial\overline{w_j}}$ applied to components of g. Thus the entries of $\overline{D^m}\widehat{\Omega_\ell^{\ell+1}}(w)$ consist of $m+\ell$ derivatives distributed among components of g. Using the factorization of $\Omega_\ell^{\ell+1}$ in (5.4), we obtain the corresponding factorization for $\widehat{\Omega_\ell^{\ell+1}}$:

$$\Omega_0^1\wedge\bigwedge_{i=1}^{\ell}\widehat{\Omega_0^1}=-\frac{1}{\ell+1}\widehat{\Omega_\ell^{\ell+1}},\tag{8.9}$$

where $\Omega_0^1=\left(\frac{\overline{g_i}}{|g|^2}\right)_{i=1}^\infty$ and $\widehat{\Omega_0^1}=\left(\frac{\overline{Dg_i}}{|g|^2}\right)_{i=1}^\infty$.

It is important for this purpose of using Lemma 10 and Remark 7 to first apply the integration by parts Lemma 3 to temper the singularity due to negative powers

of $\triangle(w,z)$, and to use the integration by parts Corollary 3 to infuse enough powers of $\left(1-|w|^2\right)$ for use in the subsequent iterated integral.

Finally it follows from Lemma 7, Proposition 2 and Proposition 3 together with the factorization (5.4) that

(8.10)
$$\left\|\left(1-|z|^2\right)^\sigma \mathcal{X}^m\widehat{\Omega_\mu^{\mu+1}}h(z)\right\|_{L^p(\lambda_n;\ell^2)} \leq C\left\|\mathbb{M}_g\right\|^{m+\mu}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

We defer the proof of (8.10) until Subsubsection 8.1.1 when further calculations are available.

Remark 8. *At this point we observe from (8.1) that the exponent $m+\mu$ in (8.10) is at most m_n+n, and thus we may take $\kappa=m_n+n$. We leave it to the interested reader to estimate the size of m_n.*

Taking into account all of the above, the conclusion is that with $\kappa=m_n+n$,
$$\|f\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)}\leq C_{n,\sigma,p,\delta}\left\|\mathbb{M}_g\right\|^{\kappa}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\|h\|_{B_p^\sigma(\mathbb{B}_n)}.$$

As the arguments described above are rather complicated we illustrate them by considering the three cases $\mu=0,1,2$ in complete detail in the next subsection before proceeding to the general case.

8.1. **Estimates in special cases.** Here we prove the estimates (8.1) for $\mu=0,1,2$. Recall that
$$\begin{aligned}\mathcal{F}^0 &= \Omega_0^1 h,\\ \mathcal{F}^1 &= \Lambda_g\mathcal{C}^{0,0}_{n,s_1}\Omega_1^2 h,\\ \mathcal{F}^2 &= \Lambda_g\mathcal{C}^{0,0}_{n,s_1}\Lambda_g\mathcal{C}^{0,1}_{n,s_2}\Omega_2^3 h.\end{aligned}$$
To obtain the estimate for $\mathcal{F}^0$ we use the multilinear inequality in Proposition 3.

In estimating $\mathcal{F}^1$ we confront for the first time a *rogue* factor $\overline{z_k-w_k}$ that we must associate with a derivative $\frac{\partial}{\partial \overline{w_k}}$ occurring in each surviving summand of the k^{th} component of the form Ω_1^2. After applying the integration by parts formula in 5 as in [20], we use the crucial inequalities in Proposition 4 and the Schur type operator estimates in Lemma 10 with $c=0$ to obtain the desired estimates. Finally we must also deal with the boundary terms in the integration by parts formula for ameliorated Charpentier kernels in Lemma 5. This requires using the radial derivative integration by parts formula in Corollary 3 as in [20], and also requires dealing with the corresponding *rogue* factors.

The final trick in the proof arises in estimating $\mathcal{F}^2$. This time there are two iterated integrals each with a *rogue* factor. The problematic *rogue* factor $\overline{z_k-\zeta_k}$ occurs in the *first* of the iterated integrals since there is *no* derivative $\frac{\partial}{\partial\overline{\zeta_k}}$ hitting the second iterated integral with which to associate the *rogue* factor $\overline{z_k-\zeta_k}$. Instead we decompose the factor as $\overline{z_k-w_k}-\overline{\zeta_k-w_k}$ and associate each of these summands with a derivative $\frac{\partial}{\partial\overline{w_k}}$ already occurring in Ω_2^3. Then we can apply the crucial inequality (6.9) and use the fact that $\sqrt{\triangle(w,z)}$ is a quasimetric to redistribute the estimates appropriately. As a result of this redistribution we are forced to use Lemma 10 with $c=\pm1$ this time as well as $c=0$. In applying the Schur type estimates in Lemma 10 to the *second* iterated integral, we require a sufficiently large

power of $\left(1-|w|^2\right)$ to be carried over from the first iterated integral. To ensure this we again use the radial derivative integration by parts formula in Corollary 3.

The estimate (8.1) for general μ involves no new ideas. There are now μ rogue terms and we need to apply Lemma 10 with $c = 0, \pm 1, ..., \pm(\mu-1)$. With this noted the arguments needed are those used above in the cases $\mu = 0, 1, 2$.

8.1.1. *The estimate for $\mathcal{F}^0$.* We begin with the estimate

$$\begin{aligned} \left\|\mathcal{F}^0\right\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} &= \left\|\Omega^1_0 h\right\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} \\ &\leq C_{n,\sigma,p,\delta} \left\|\mathbb{M}_g\right\|^m_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|_{B^\sigma_{p,m}(\mathbb{B}_n)}, \end{aligned}$$

for $m+\sigma > \frac{n}{p}$. However, for later use we prove instead the more general estimate with $\mathcal{X}$ in place of R, except that m must then be chosen twice as large:

$$\begin{aligned} &\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^\sigma \mathcal{X}^m \left(\Omega^1_0 h\right)(z)\right|^p d\lambda_n(z) \\ &\qquad \leq C_{n,\sigma,p,\delta} \left\|\mathbb{M}_g\right\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}, \end{aligned} \tag{8.11}$$

for $m > 2\left(\frac{n}{p}-\sigma\right)$. Recall that $\mathcal{X}^m$ is the differential operator of order m given in Definition 7 that is adapted to the complex geometry of the unit ball $\mathbb{B}_n$. It will be in estimating iterated Charpentier integrals below that the derivatives R^m and $\overline{\mathcal{D}^m}$ will arise from integration by parts in the previous iterated integral, and this will require estimates using $\mathcal{X}^m$.

By Leibniz' rule for $\mathcal{X}^m$ we have

$$\mathcal{X}^m\left(\Omega^1_0 h\right) = \sum_{k=0}^m c_k \left(\mathcal{X}^k \Omega^1_0\right)\left(\mathcal{X}^{m-k} h\right),$$

and

$$\mathcal{X}^k\left(\Omega^1_0\right) = \mathcal{X}^k\left(\frac{\overline{g}}{|g|^2}\right) = \sum_{\ell=0}^k c_\ell \left(\mathcal{X}^{k-\ell}\overline{g}\right)\left(\mathcal{X}^\ell |g|^{-2}\right). \tag{8.12}$$

It suffices to prove

$$\begin{aligned} &\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^\sigma \left(\sum_{k=0}^m \sum_{\ell=0}^k c_k c_\ell \left(\mathcal{X}^{k-\ell}\overline{g}\right)\left(\mathcal{X}^\ell |g|^{-2}\right)\left(\mathcal{X}^{m-k} h\right)\right)\right|^p d\lambda_n \\ &\qquad \leq C_{n,\sigma,p,\delta} \left\|\mathbb{M}_g\right\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}, \end{aligned}$$

and hence

$$\begin{aligned} &\int_{\mathbb{B}_n} \left(1-|z|^2\right)^{p\sigma} \left|\mathcal{X}^{k-\ell}\overline{g}\right|^p \left|\mathcal{X}^\ell |g|^{-2}\right|^p \left|\mathcal{X}^{m-k} h\right|^p d\lambda_n \\ &\qquad \leq C_{n,\sigma,p,\delta} \left\|\mathbb{M}_g\right\|^{mp}_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \|h\|^p_{B^\sigma_p(\mathbb{B}_n)}, \end{aligned} \tag{8.13}$$

for each fixed $0 \leq \ell \leq k \leq m$.

Now we can profitably estimate both $\left|\mathcal{X}^{m-k} h\right|$ and $\left|\mathcal{X}^{k-\ell}\overline{g}\right|$ as they are, but we must be more careful with $\left|\mathcal{X}^\ell |g|^{-2}\right|$. In the case $\ell = 1$, we assume for convenience

that $\mathcal{X}$ annihilates g_i (if not it will annihilate $\overline{g_i}$ unless $\mathcal{X}=I$) and obtain,

$$\begin{aligned}\left|\mathcal{X}\left|g\right|^{-2}\right|^{2} &= \left|-\left|g\right|^{-4}\sum_{i=1}^{\infty}g_i\mathcal{X}\overline{g_i}\right|^{2}\\ &\leq \left|g\right|^{-8}\left(\sum_{i=1}^{\infty}\left|g_i\right|^{2}\right)\left(\sum_{i=1}^{\infty}\left|\mathcal{X}\overline{g_i}\right|^{2}\right)\leq\left|g\right|^{-6}\sum_{i=1}^{\infty}\left|\mathcal{X}\overline{g_i}\right|^{2}.\end{aligned}$$

Similarly when $\ell=2$,

$$\begin{aligned}\left|\mathcal{X}^2\left|g\right|^{-2}\right|^{2} &= \left|-\left|g\right|^{-4}\sum_{i=1}^{\infty}g_i\mathcal{X}^2\overline{g_i}+2\left|g\right|^{-6}\sum_{i\neq j}\left(g_i\mathcal{X}\overline{g_i}\right)\left(g_j\mathcal{X}\overline{g_j}\right)\right|^{2}\\ &\leq 2\left|g\right|^{-6}\sum_{i=1}^{\infty}\left|\mathcal{X}^2\overline{g_i}\right|^{2}+4\left|g\right|^{-8}\left(\sum_{i=1}^{\infty}\left|\mathcal{X}\overline{g_i}\right|^{2}\right)^{2},\end{aligned}$$

and the general case is

$$\begin{aligned}&\left|\mathcal{X}^{\ell}\left|g\right|^{-2}\right|^{2} \qquad (8.14)\\ &\leq C_{\ell}\left|g\right|^{-6}\sum_{i=1}^{\infty}\left|\mathcal{X}^{\ell}g_i\right|^{2}+C_{\ell-1}\left|g\right|^{-8}\left(\sum_{i=1}^{\infty}\left|\mathcal{X}^{\ell-1}\overline{g_i}\right|^{2}\right)\left(\sum_{i=1}^{\infty}\left|\mathcal{X}\overline{g_i}\right|^{2}\right)\\ &\quad+\ldots+C_0\left|g\right|^{-4-2\ell}\left(\sum_{i=1}^{\infty}\left|\mathcal{X}\overline{g_i}\right|^{2}\right)^{\ell}\\ &=\sum_{1\leq\alpha_1\leq\alpha_2\leq\ldots\leq\alpha_M:\alpha_1+\alpha_2+\ldots+\alpha_M=\ell}c_{\alpha}\left|g\right|^{-4-2\ell}\prod_{m=1}^{M}\left(\sum_{i=1}^{\infty}\left|\mathcal{X}^{\alpha_m}\overline{g_i}\right|^{2}\right).\end{aligned}$$

We can ignore the powers of $|g|$ since $|g|$ is bounded above and below by Lemma 8 and the hypotheses of Theorem 2. Fixing α we see that the left side of (8.13) is thus at most

$$C_{n,\sigma,p,\delta}\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{p\sigma}\left|\mathcal{X}^{k-\ell}\overline{g}\right|^{p}\left|\mathcal{Y}^{m-k}h\right|^{p}\left(\prod_{j=1}^{M}\left|\mathcal{X}^{\alpha_j}\overline{g}\right|^{p}\right)d\lambda_n.$$

Since $\left|\mathcal{X}^{k-\ell}\overline{g}\right|^{2}=\sum_{i=1}^{\infty}\left|\mathcal{X}^{k-\ell}\overline{g_i}\right|^{2}$ and $k-\ell$ could vanish (unlike the exponents α_ℓ which are positive), we see that altogether after renumbering, it suffices to prove

$$\begin{aligned}&\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{p\sigma}\left|\mathcal{Y}^{\alpha_1}h\right|^{p}\left|\mathcal{Y}^{\alpha_2}g\right|^{p}\ldots\left|\mathcal{Y}^{\alpha_M}g\right|^{p}d\lambda_n \qquad (8.15)\\ &\qquad\leq C_{n,\sigma,p,\delta}\left\|\mathbb{M}_g\right\|^{Mp}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\left\|h\right\|^{p}_{B_p^\sigma(\mathbb{B}_n)}\end{aligned}$$

for each fixed $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_M)$ with $M\geq 2$, $|\alpha|=m$ and at most one of $\alpha_2,\ldots,\alpha_M$ is zero. We have used here that $\left|\overline{D}\overline{g}\right|=|Dg|$. Now Proposition 3 yields (8.15) for each $0\leq k\leq m$ and $|\alpha|=m-k$. Summing these estimates completes the proof of (8.11).

We can now prove the more general inequality (8.10). Indeed, using the factorization (5.4) of $\widehat{\Omega_\mu^{\mu+1}}$ together with the Leibniz formula gives

$$\begin{aligned}\mathcal{X}^m\left(\widehat{\Omega_\mu^{\mu+1}}h\right) &= \mathcal{X}^m\left(\Omega_0^1\wedge\left(\widehat{\Omega_0^1}\right)^\mu h\right)\\ &= \sum_{\alpha\in\mathbb{Z}_+^{\mu+2}:|\alpha|=m}\left(\mathcal{X}^{\alpha_0}\Omega_0^1\right)\wedge\bigwedge_{j=1}^{\mu}\left(\mathcal{X}^{\alpha_j}\widehat{\Omega_0^1}\right)\left(\mathcal{X}^{\alpha_{\mu+1}}h\right)\\ &= \sum_{\alpha\in\mathbb{Z}_+^{\mu+2}:|\alpha|=m}\left\{\left(\mathcal{X}^{\alpha_0}\Omega_0^1\right)\wedge\bigwedge_{j=1}^{\mu}\left(\mathcal{X}^{\alpha_j+1}\Omega_0^1\right)\right\}\left(\mathcal{X}^{\alpha_{\mu+1}}h\right),\end{aligned}$$

where we have used that $\widehat{\Omega_0^1}$ already has an $\mathcal{X}$ derivative in each summand, and so $\mathcal{X}^{\alpha_j}\widehat{\Omega_0^1}$ can be written as $\mathcal{X}^{\alpha_j+1}\Omega_0^1$. Now use (8.12) and (8.14) to see that $\left|\mathcal{X}^m\left(\widehat{\Omega_\mu^{\mu+1}}h\right)\right|$ is controlled by a tensor product of at most $m+\mu$ factors, and then apply Proposition 3 as above to complete the proof of (8.10).

8.1.2. *The estimate for $\mathcal{F}^1$.* The estimate in (8.1) with $\mu=1$ will follow from (8.10) and the estimate

$$\begin{aligned}&\left\|\left(1-|z|^2\right)^\sigma\mathcal{Y}^{m_1}\left(\Lambda_g\mathcal{C}_{n,s}^{0,0}\Omega_1^2h\right)\right\|_{L^p(\lambda_n)}^p\\ &\qquad\le C\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\mathcal{X}^{m_2}\left(\widehat{\Omega_1^2}h\right)(z)\right|^p d\lambda_n(z),\end{aligned}\tag{8.16}$$

where as in Definition 9, we define $\widehat{\Omega_1^2}$ to be Ω_1^2 with ∂ replaced by D throughout:

$$\widehat{\Omega_1^2}=\sum_{j,k=1}^N\frac{\overline{\{g_kDg_j-g_jDg_k\}}}{|g|^4}e_j\wedge e_k,$$

and where $Dh=\sum_{k=1}^n(D_kh)\,dz_k$ and D_k is the k^{th} component of D. We are using here the following observation regarding the interior product $\Omega_1^2h\lrcorner d\overline{w_k}$:

(8.17) For each summand of $\Omega_1^2h\lrcorner d\overline{w_k}$, there is a unique $1\le i\le N$ so that $\dfrac{\partial g_i}{\partial\overline{w_k}}$ occurs as a factor in the summand.

We rewrite (8.16) as

$$\begin{aligned}&\left\|\left(1-|z|^2\right)^\sigma\mathcal{R}^{m_1''}D^{m_1'}\left(\Lambda_g\mathcal{C}_{n,s}^{0,0}\Omega_1^2h\right)\right\|_{L^p(\lambda_n)}^p\\ &\qquad\le C\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\mathcal{R}^{m_2''}\overline{D^{m_2'}}\left(\widehat{\Omega_1^2}h\right)(z)\right|^p d\lambda_n(z),\end{aligned}\tag{8.18}$$

where $\mathcal{R}^m=\left(1-|z|^2\right)^m\left(R^k\right)_{k=0}^m$ as in (8.2). As mentioned above, we only need to prove the case $m_1''=0$ since (8.1) only requires that we estimate $\left\|\mathcal{F}^1\right\|_{B_{p,m}^\sigma(\mathbb{B}_n)}$. However, when considering the estimate for $\mathcal{F}^2$ in (8.1) we will no longer have the luxury of using the norm $\|\cdot\|_{B_{p,m}^\sigma(\mathbb{B}_n)}$ in the second iterated integral occuring there, and so we will consider the more general case now in preparation for what comes later. As we will see however, it is necessary to choose m_1' sufficiently large in order

to obtain (8.18). It is useful to recall that the operator $\left(1-|z|^2\right)R$ is "smaller" than $\overline{D}$ in the sense that

$$\begin{aligned}\overline{D} &= \left(1-|z|^2\right)P_z\overline{\nabla}+\sqrt{1-|z|^2}Q_z\overline{\nabla},\\ \left(1-|z|^2\right)R &= \left(1-|z|^2\right)P_z\nabla.\end{aligned}$$

To prove (8.18) we will ignore the contraction Λ_g since if derivatives hit g in the contraction, the estimates are similar if not easier. Note also that $|\Lambda_g F|\leq |g|\,|F|$ for the contraction $\Lambda_g F$ of any tensor F.

We will also initially suppose that $m_1''=0$ and later take m_1'' sufficiently large. Now we apply Lemma 5 to $\mathcal{C}_{n,s}^{0,0}\Omega_1^2 h$ and obtain

$$\begin{aligned}\mathcal{C}_{n,s}^{0,0}\Omega_1^2 h(z) &= c_0\mathcal{C}_{n,s}^{0,0}\left(\overline{\mathcal{D}}^{m_2'}\Omega_1^2 h\right)(z)+\textit{boundary terms}\\ &= \int_{\mathbb{B}_n}\Phi_{n,s}^0(w,z)\,\overline{\mathcal{D}}^{m_2'}\left(\Omega_1^2 h\right)dV(w)\\ &\quad+\textit{boundary terms}.\end{aligned}\tag{8.19}$$

A typical term above looks like

$$\int_{\mathbb{B}_n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\frac{(1-w\overline{z})^{n-1}}{\triangle(w,z)^n}\overline{\mathcal{D}}^{m_2'}\left(\Omega_1^2 h\right)dV(w)\tag{8.20}$$

where we are discarding the sum of (balanced) factors $\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\right)^j$ for $1\leq j\leq n-1$ in Lemma 5 that turn out to only help with the estimates. This can be seen from (6.11) and its trivial counterpart

$$\left|D_{(z)}^m\left\{\left(1-|z|^2\right)^k\right\}\right|+\left|\left(1-|z|^2\right)^m R_{(z)}^m\left\{\left(1-|z|^2\right)^k\right\}\right|\leq C\left(1-|z|^2\right)^k.$$

Recall from the general discussion above that in the integral (8.20) there are *rogue* factors $\overline{z_k-w_k}$ in $\overline{\mathcal{D}}^{m_2'}\left(\Omega_1^2 h\right)(w)$ that must be associated with a $\frac{\partial}{\partial\overline{w_k}}$ derivative that hits some factor of each summand in the k^{th} component $\Omega_1^2\lrcorner d\overline{w_k}$ of $\Omega_1^2\approx\overline{\{g_i\partial g_j-g_j\partial g_i\}}$. Thus we can apply (6.9) to the components of $\Omega_1^2 h(z)$ to obtain

$$\begin{aligned}&\left|\overline{\mathcal{D}}^{m_2'}\Omega_1^2 h(z)\right|\\ \approx&\left|\sum_{k=1}^n\sum_{|\alpha|=m_2'}\left(\overline{w_k}-\overline{z_k}\right)\left(\overline{w-z}\right)^\alpha\frac{\partial^{m_2'}}{\partial\overline{w}^\alpha}\left(\Omega_1^2 h\lrcorner d\overline{w_k}\right)\right|\\ \leq&\ C\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D^{m_2'}}\left(\widehat{\Omega_1^2}h\right)(w)\right|.\end{aligned}\tag{8.21}$$

Thus we get

$$\begin{aligned}
(8.22)\qquad & \left(1-|z|^2\right)^\sigma \left|D^{m_1'}\mathcal{C}_{n,s}^{0,0}\Omega_1^2 h(z)\right| \\
\leq & \int_{\mathbb{B}_n} \left(1-|z|^2\right)^\sigma \left| D_{(z)}^{m_1'}\left\{\frac{\left(1-|w|^2\right)^{s-n}(1-w\overline{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n}\right\}\right| \\
& \times\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{m_2'+1}\left|\overline{D^{m_2'}}\left(\widehat{\Omega_1^2}h\right)(w)\right| dV(w) \\
\equiv & \ S_{m_1',m_2'}^s f(z),
\end{aligned}$$

where

$$f(w)=\left(1-|w|^2\right)^\sigma\left|\overline{D^{m_2'}}\left(\widehat{\Omega_1^2}h\right)(w)\right|. \tag{8.23}$$

Now we iterate the estimate (6.10),

$$\left|D_{(z)}\triangle(w,z)\right|\leq C\left(1-|z|^2\right)\triangle(w,z)^{\frac{1}{2}}+\triangle(w,z),$$

to obtain

$$\begin{aligned}
(8.24)\qquad & \left| D_{(z)}^{m_1'}\left\{\frac{\left(1-|w|^2\right)^{s-n}(1-w\overline{z})^{n-1}}{(1-\overline{w}z)^{s-n}\triangle(w,z)^n}\right\}\right| \\
\leq & \ \frac{\left(1-|z|^2\right)^{m_1'}\left(1-|w|^2\right)^{s-n}\triangle(w,z)^{\frac{m_1'}{2}}}{|1-w\overline{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}} \\
& +\ldots+\frac{\left(1-|w|^2\right)^{s-n}}{|1-w\overline{z}|^{s-2n+1}\triangle(w,z)^n}+OK,
\end{aligned}$$

where the terms in OK are obtained when some of the derivatives D hit the factor $\frac{1}{(1-\overline{w}z)^{s-n}}$ or factors $D\triangle(w,z)$ already in the numerator. Leaving the OK terms for later, we combine all the estimates above to get that if we plug the first term on the right in (8.24) into the left side of (8.18), then the result is dominated by

$$\begin{aligned}
& \int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{m_1'+\sigma}\left(1-|w|^2\right)^{s-n-m_2'-1-\sigma}\triangle(w,z)^{\frac{m_1'+m_2'+1}{2}}}{|1-w\overline{z}|^{s-2n+1}\triangle(w,z)^{n+m_1'}}f(w)\,dV(w) \\
= & \int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{m_1'+\sigma}\left(1-|w|^2\right)^{s-n-1-m_2'-\sigma}}{|1-w\overline{z}|^{s-2n+1}}\sqrt{\triangle(w,z)}^{m_2'-m_1'-2n+1}f(w)\,dV(w).
\end{aligned}$$

Now for convenience choose $m_2'=m_1'+2n-1$ so that the factor of $\sqrt{\triangle(w,z)}$ disappears. We then get

(8.25)

$$\left(1-|z|^2\right)^\sigma\left|D^{m_1'}\mathcal{C}_{n,s}^{0,0}\Omega_1^2h(z)\right|\leq\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{m_1'+\sigma}\left(1-|w|^2\right)^{s-3n-m_1'-\sigma}}{|1-w\overline{z}|^{s-2n+1}}f(w)\,dV(w).$$

Lemma 10 shows that the operator

$$T_{a,b,0}f(z) = \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^a \left(1-|w|^2\right)^b}{|1-w\overline{z}|^{n+1+a+b}} f(w)\, dV(w)$$

is bounded on $L^p\left(\mathbb{B}_n; \left(1-|w|^2\right)^t dV(w)\right)$ if and only if

$$-pa < t+1 < p(b+1).$$

We apply this lemma with $t = -n-1$, $a = m_1' + \sigma$ and $b = s - 3n - m_1' - \sigma$. Note that the sum of the exponents in the numerator and denominator of (8.25) are equal if we write the integral in terms of invariant measure $d\lambda_n(w) = \left(1-|w|^2\right)^{-n-1} dV(w)$. We conclude that $S^s_{m_1',m_2'}$ is bounded on $L^p(d\lambda_n)$ provided T is, and that this latter happens if and only if

$$-p(m_1'+\sigma) < -n < p(s - 3n + 1 - m_1' - \sigma).$$

This requires $m_1' + \sigma > \frac{n}{p}$ and $s > 3n - 1 + m_1' + \sigma - \frac{n}{p}$.

Remark 9. *Suppose instead that we choose m_2' above to be a positive integer satisfying $c = m_2' - m_1' - 2n + 1 > -2n$. Then we would be dealing with the operator $T_{a,b,c}$ where $a = m_1' + \sigma$ and*

$$b = s - n - 1 - m_2' - \sigma = s - 3n - c - m_1' - \sigma.$$

By Lemma 10, $T_{a,b,c}$ is bounded on $L^p(d\lambda_n)$ if and only if

$$-p(m_1'+\sigma) < -n < p(s - 3n + 1 - c - m_1' - \sigma),$$

i.e. $m_1' + \sigma > \frac{n}{p}$ and $s > c + 3n - 1 + m_1' + \sigma - \frac{n}{p}$. Thus we can use any *value of $c > -2n$ provided we choose $m_2' \geq m_1'$ and s large enough.*

Now we turn to the second displayed term on the right side of (8.24) which leads to the operator $T_{a,b,0}$ with $a = \sigma$, $b = s - 3n - \sigma$. This time we will *not* in general have the required boundedness condition $\sigma > \frac{n}{p}$. It is for this reason that we must return to (8.18) and insist that m_1'' be chosen sufficiently large that $m_1'' + \sigma > \frac{n}{p}$. For convenience we let $m_1' = 0$ for now. Indeed, it follows from the second line in the crucial inequality (6.10) that the second displayed term on the right side of (8.24) is

$$\frac{\left(1-|z|^2\right)^{m_1''} \left(1-|w|^2\right)^{s-n} \triangle(w,z)^{\frac{m_1''}{2}}}{|1-w\overline{z}|^{s-2n+1} \triangle(w,z)^{n+m_1''}} + \textit{better terms.}$$

Using this expression and choosing $m_2' = m_1'' + 2n - 1$ so that the term $\sqrt{\triangle(w,z)}$ disappears from the ensuing integral, we obtain the following analogue of (8.25):

$$\begin{aligned} &\left(1-|z|^2\right)^{\sigma} \left(1-|z|^2\right)^{m_1''} \left|\mathcal{R}^{m_1''} \mathcal{C}^{0,0}_{n,s} \Omega^2_1 h(z)\right| \\ \leq &\int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{m_1''+\sigma} \left(1-|w|^2\right)^{s-3n-m_1''-\sigma}}{|1-w\overline{z}|^{s-2n+1}} f(w)\, dV(w). \end{aligned}$$

The corresponding operator $T_{a,b,0}$ has $a = m_1'' + \sigma$ and $b = s - 3n - m_1'' - \sigma$ and is bounded on $L^p(\lambda_n)$ when $-p(m_1''+\sigma) < -n < p(s - 3n + 1 - m_1'' - \sigma)$. Thus

there is no unnecessary restriction on σ if m_1'' and s are chosen appropriately large. Note that the only difference between this operator $T_{a,b,0}$ and the previous one is that m_1' has been replaced by m_1''.

The above arguments are easily modified to handle the general case of (8.18) provided $m_1''+\sigma>\frac{n}{p}$ and s is chosen sufficiently large.

Now we return to consider the OK terms in (8.24). For this we use the inequality (6.11):

$$\left|D_{(z)}^{m}\left\{(1-\overline{w}z)^k\right\}\right|\leq C\left|1-\overline{w}z\right|^k\left(\frac{1-|z|^2}{|1-\overline{w}z|}\right)^{\frac{m}{2}}.$$

We ignore the derivative $\left(1-|z|^2\right)R$ as the second line in (6.11) shows that it satisfies a better estimate. We also write m_1 and m_2 in place of m_1' and m_2' now. As a result, one of the extremal OK terms in (8.24) is

$$\frac{\left(1-|z|^2\right)^{\frac{m_1}{2}}\left(1-|w|^2\right)^{s-n}}{|1-w\overline{z}|^{s-2n+1+\frac{m_1}{2}}\triangle(w,z)^n},$$

which when combined with the other estimates leads to the integral operator

$$\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{\frac{m_1}{2}+\sigma}\left(1-|w|^2\right)^{s-n-1-m_2-\sigma}}{|1-w\overline{z}|^{s-2n+1+\frac{m_1}{2}}}\sqrt{\triangle(w,z)}^{m_2-2n-1}f(w)\,dV(w).$$

This is $T_{a,b,c}$ with $a=\frac{m_1}{2}+\sigma$, $b=s-n-1-m_2-\sigma$ and $c=m_2-2n-1$. This is bounded on $L^p(\lambda_n)$ provided $m_2\geq 2$ and

$$-p\left(\frac{m_1}{2}+\sigma\right)<-n<p\left(s-n-m_2-\sigma\right),$$

i.e. $\frac{m_1}{2}+\sigma>\frac{n}{p}$ and $s>n+m_2+\sigma-\frac{n}{p}$. The intermediate OK terms are handled similarly. Note that the crux of the matter is that all of the positive operators have the form $T_{a,b,c}$, and moreover, if s and the $m's$ are chosen appropriately large, then $T_{a,b,c}$ is bounded on $L^p(\lambda_n)$.

8.1.3. *Boundary terms for $\mathcal{F}^1$.* Now we turn to estimating the boundary terms in (8.19). A typical term is

$$(8.26)\quad \mathcal{S}_{n,s}\left(\overline{\mathcal{D}}^k\left(\Omega_1^2h\right)\right)\left[\overline{\mathcal{Z}}\right](z)=\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{s-n-1}}{(1-\overline{w}z)^s}\overline{\mathcal{D}}^k\left(\Omega_1^2h\right)\left[\overline{\mathcal{Z}}\right](w)\,dV(w),$$

with $0\leq k\leq m-1$ upon appealing to Lemma 5.

We now apply the operator $\left(1-|z|^2\right)^{m_1+\sigma}R^{m_1}$ to the integral in the right side of (8.26) and using Proposition 4 we obtain that the absolute value of the result is dominated by

$$\begin{aligned}&\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{m_1+\sigma}\left(1-|w|^2\right)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\left|\overline{D}^k\left(\widehat{\Omega_1^2h}\right)\right|dV(w)\\ =&\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{m_1+\sigma}\left(1-|w|^2\right)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{k+1}}{|1-\overline{w}z|^{s+m_1}}\left|\left(1-|w|^2\right)^{\sigma}\overline{D}^k\left(\widehat{\Omega_1^2h}\right)(w)\right|dV(w).\end{aligned}$$

The operator in question here is $T_{a,b,c}$ with $a = m_1 + \sigma$, $b = s - n - 2 - k - \sigma$ and $c = k + 1$ since

$$a + b + c + n + 1 = s + m_1.$$

Lemma 10 applies to prove the desired boundedness on $L^p(\lambda_n)$ provided $m_1 + \sigma > \frac{n}{p}$.

However, if k fails to satisfy $k + 1 > 2\left(\frac{n}{p} - \sigma\right)$, then the derivative $D^{k+1}\Omega$ cannot be used to control the norm $\|\Omega\|_{B_p^\sigma(\mathbb{B}_n)}$. To compensate for a small k, we must then apply Corollary 3 to the right side of (8.26) (which for fixed z is in $C\left(\overline{\mathbb{B}_n}\right) \cap C^\infty(\mathbb{B}_n)$) before differentiating and taking absolute values inside the integral. This then leads to operators of the form

$$\left(1-|z|^2\right)^{m_1+\sigma} R^{m_1}\left\{\int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^{s-n-1}}{(1-\overline{w}z)^s} \right.$$
$$\left. \times \left(1-|w|^2\right)^m R^m\left[\overline{\mathcal{D}}^k\left(\Omega_1^2 h\right)(w)\right] dV(w)\right\},$$

which are dominated by

$$\int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{m_1+\sigma}\left(1-|w|^2\right)^{s-n-1}}{|1-\overline{w}z|^{s+m_1}}$$
$$\times \left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)^{k+1}\left|\mathcal{R}^m\overline{D}^k\left(\widehat{\Omega_1^2}h\right)(w)\right| dV(w),$$

which is

$$\int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{m_1+\sigma}\left(1-|w|^2\right)^{s-n-2-k-\sigma}\sqrt{\triangle(w,z)}^{k+1}}{|1-\overline{w}z|^{s+m_1}}$$
$$\times \left|\left(1-|w|^2\right)^\sigma \mathcal{R}^m\overline{D}^k\left(\widehat{\Omega_1^2}h\right)(w)\right| dV(w).$$

This latter operator is $T_{a,b,c}H(z)$ with

$$a = m_1 + \sigma, b = s - n - 2 - k - \sigma, c = k + 1$$

and $H(w) = \left|\left(1-|w|^2\right)^\sigma R_{b'}^m\overline{D}^k\left(\widehat{\Omega_1^2}h\right)(w)\right|$. Note that for $m > 2\left(\frac{n}{p} - \sigma\right)$ we do indeed now have $\|H\|_{L^p(\lambda_n)} \approx \left\|\widehat{\Omega_1^2}h\right\|_{B_p^\sigma(\mathbb{B}_n)}$. The operator here is the same as that above and so Lemma 10 applies to prove the desired boundedness on $L^p(\lambda_n)$.

8.1.4. *The estimate for $\mathcal{F}^2$.* Our next task is to obtain the estimate (8.1) for $\mu = 2$, and for this we will show that

$$\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m_1+\sigma} R^{m_1}\Lambda_g\mathcal{C}_{n,s_1}^{0,0}\Lambda_g\mathcal{C}_{n,s_2}^{0,1}\Omega_2^3\right|^p d\lambda_n(z) \tag{8.27}$$
$$\leq C\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\left(1-|z|^2\right)^{m_3''} R^{m_3''}\overline{D}^{m_3'}\left(\widehat{\Omega_2^3}h\right)(z)\right|^p d\lambda_n(z).$$

Unlike the previous argument we will have to deal with a *rogue* term $\left(\overline{z_2} - \overline{\xi_2}\right)$ this time where there is no derivative $\frac{\partial}{\partial\overline{\xi_2}}$ to associate to the factor $\left(\overline{z_2} - \overline{\xi_2}\right)$. Again we

ignore the contractions Λ_g. Then we use Lemma 5 to perform integration by parts m_2' times in the first iterated integral and m_3' times in the second iterated integral. We also use Corollary 3 to perform integration by parts in the *radial* derivative m_2'' times in the *first* iterated integral (for fixed z, $\mathcal{C}_{n,s_2}^{0,1}\Omega_2^3 \in C\left(\overline{\mathbb{B}_n}\right) \cap C^\infty\left(\mathbb{B}_n\right)$ by standard estimates [13]), so that the additional factor $\left(1-|\xi|^2\right)^{m_2''}$ can be used crucially in the second iterated integral, and also m_3'' times in the *second* iterated integral for use in acting on Ω_2^3.

Recall from Lemma 5 that

$$\begin{aligned}\mathcal{C}_{n,s}^{0,q}\eta\left(z\right) &= \textit{boundary terms}\text{ (depending on } m\text{)}\\ &+\sum_{\ell=0}^{q}\int_{\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-1-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle\left(w,z\right)^n}\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{s-n}\\ &\times\left(\sum_{j=0}^{n-\ell-1}c_{j,\ell,n,s}\left[\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right]^j\right)\overline{\mathcal{D}}^m\eta\left(z\right).\end{aligned}$$

Recall also that that $\overline{\mathcal{D}}^m$ already has the rogue terms built in, as can be seen from (4.6). Now we use the right side above with $q=\ell=j=0$ to substitute for $\mathcal{C}_{n,s_1}^{0,0}$, and the right side above with $q=\ell=1$ and $j=0$ to substitute for $\mathcal{C}_{n,s_2}^{0,1}$. Then a typical part of the resulting kernel of the operator $\mathcal{C}_{n,s_1}^{0,0}\mathcal{C}_{n,s_2}^{0,1}\Omega_2^3\left(z\right)$ is

$$\begin{aligned}&\int_{\mathbb{B}_n}\frac{\left(1-\xi\overline{z}\right)^{n-1}}{\triangle\left(\xi,z\right)^n}\left(\frac{1-|\xi|^2}{1-\overline{\xi}z}\right)^{s_1-n}\left(\overline{z_2}-\overline{\xi_2}\right)\\ &\times\left(1-|\xi|^2\right)^{m_2'}R^{m_2'}\overline{\mathcal{D}}^{m_2''}\int_{\mathbb{B}_n}\frac{\left(1-w\overline{\xi}\right)^{n-2}\left(1-|w|^2\right)}{\triangle\left(w,\xi\right)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{s_2-n}\\ &\times\left(\overline{w_1}-\overline{\xi_1}\right)\left(1-|w|^2\right)^{m_3'}R^{m_3'}\overline{\mathcal{D}}^{m_3''}\left(\Omega_2^3h\right)\left(w\right)dV\left(w\right)dV\left(\xi\right),\end{aligned}\tag{8.28}$$

where we have arbitrarily chosen $\left(\overline{z_2}-\overline{\xi_2}\right)$ and $\left(\overline{w_1}-\overline{\xi_1}\right)$ as the *rogue* factors.

Remark 10. *It is important to note that the differential operators $\overline{\mathcal{D}_\zeta^{m_2}}$ are conjugate in the variable z and hence* vanish *on the kernels of the boundary terms $\mathcal{S}_{n,s}\left(\overline{\mathcal{D}}^k\Omega_2^3h\right)(z)$ in the integration by parts formula (4.7) associated to the Charpentier solution operator $\mathcal{C}_{n,s_2}^{0,1}$ since these kernels are* holomorphic. *As a result the operator $\overline{\mathcal{D}}^{m_2'}$ hits only the factor $\overline{\mathcal{D}}^k\Omega_2^3h$ and a typical term is*

$$\overline{(z_i-\zeta_i)}\frac{\partial}{\partial\overline{z_i}}\left\{\overline{(w_i-z_i)}\Omega_2^3h\right\}=-\overline{(z_i-\zeta_i)}\Omega_2^3h,$$

where the derivative $\frac{\partial}{\partial\overline{w_i}}$ must occur in each surviving term in Ω_2^3h, and this term which is then handled like the rogue *terms.*

Now we recall the factorization (5.4) with $\ell=2$,

$$\Omega_2^3=-4\Omega_0^1\wedge\widetilde{\Omega_0^1}\wedge\widetilde{\Omega_0^1},$$

and that $\Omega_2^3(w)$ must have both derivatives $\frac{\partial g}{\partial \overline{w_1}}$ and $\frac{\partial g}{\partial \overline{w_2}}$ occurring in it, one surviving in each of the factors $\widetilde{\Omega_0^1}$, along with other harmless powers of g that we ignore. Thus we may replace $\widetilde{\Omega_0^1} \wedge \widetilde{\Omega_0^1}$ with $\frac{\partial}{\partial \overline{w_2}}\Omega_0^1 \wedge \frac{\partial}{\partial \overline{w_1}}\Omega_0^1$. If we use

$$\overline{z_2} - \overline{\xi_2} = (\overline{z_2} - \overline{w_2}) - (\overline{\xi_2} - \overline{w_2}),$$

we can write the above iterated integral as

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-\xi\overline{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\overline{\xi}z}\right)^{s_1-n} \\
&\times \int_{\mathbb{B}_n} \left(1-|\xi|^2\right)^{m_2''} R^{m_2''}\overline{\mathcal{D}}^{m_2'}\left\{\frac{(1-w\overline{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{s_2-n}\right\} \\
&\times \left[\left(1-|w|^2\right)^{m_3''} R^{m_3''}\left(\overline{\xi_2}-\overline{w_2}\right)\frac{\partial}{\partial\overline{w_2}}\overline{\mathcal{D}}^{m_3'-\ell}\Omega_0^1\right] \\
&\wedge \left[\left(1-|w|^2\right)^{m_3''} R^{m_3''}\left(\overline{\xi_1}-\overline{w_1}\right)\frac{\partial}{\partial\overline{w_1}}\overline{\mathcal{D}}^{\ell}\Omega_0^1\right] dV(w)\, dV(\xi)
\end{aligned}$$

minus

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-\xi\overline{z})^{n-1}}{\triangle(\xi,z)^n}\left(\frac{1-|\xi|^2}{1-\overline{\xi}z}\right)^{s_1-n} \\
&\times \int_{\mathbb{B}_n} \left(1-|\xi|^2\right)^{m_2''} R^{m_2''}\overline{\mathcal{D}}^{m_2'}\left\{\frac{(1-w\overline{\xi})^{n-2}(1-|w|^2)}{\triangle(w,\xi)^n}\left(\frac{1-|w|^2}{1-\overline{w}\xi}\right)^{s_2-n}\right\} \\
&\times \left[\left(1-|w|^2\right)^{m_3''} R^{m_3''}\left(\overline{z_2}-\overline{w_2}\right)\frac{\partial}{\partial\overline{w_2}}\overline{\mathcal{D}}^{m_3'-\ell}\Omega_0^1\right] \\
&\wedge \left[\left(1-|w|^2\right)^{m_3''} R^{m_3''}\left(\overline{\xi_1}-\overline{w_1}\right)\frac{\partial}{\partial\overline{w_1}}\overline{\mathcal{D}}^{\ell}\Omega_0^1\right] dV(w)\, dV(\xi),
\end{aligned}$$

where we have temporarily ignored the wedge products with terms that do not include derivatives of g, as these terms are bounded and so harmless.

Now we apply $\left(1-|z|^2\right)^{\sigma}\left(1-|z|^2\right)^{m_1''} R^{m_1''}D^{m_1'}$ to these operators. Using the crucial inequalities in Proposition 4 together with the factorization (8.9) with $\ell = 2$,

$$\widehat{\Omega_2^3} = -4\Omega_0^1 \wedge \widehat{\Omega_0^1} \wedge \widehat{\Omega_0^1},$$

the result of this application on the first integral is then dominated by

$$(8.29)\quad \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{\sigma}|1-\xi\overline{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1''}$$
$$\times\left\{\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right\}\left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-n}$$
$$\times\int_{\mathbb{B}_n}\frac{\left(1-|\xi|^2\right)^{m_2''}\left|1-w\overline{\xi}\right|^{n-2}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+n}}\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}$$
$$\times\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2''}\left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2'}+\triangle(w,\xi)^{m_2'}\right\}$$
$$\times\left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-n}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{2}$$
$$\times\left|\left(1-|w|^2\right)^{m_3''}R^{m_3''}\overline{D^{m_3'}}\left(\widehat{\Omega_2^3}h\right)(w)\right|dV(w)\,dV(\xi),$$

and the result of this application on the second integral is dominated by

$$(8.30)\quad \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{\sigma}|1-\xi\overline{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+2}}\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1''}$$
$$\times\left\{\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right\}\left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-n}$$
$$\times\int_{\mathbb{B}_n}\frac{\left(1-|\xi|^2\right)^{m_2''}\left|1-w\overline{\xi}\right|^{n-2}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+n}}\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}$$
$$\times\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2''}\left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m}+\triangle(w,\xi)^{m_2'}\right\}$$
$$\times\left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-n}\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'}\left(\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\right)\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)$$
$$\times\left|\left(1-|w|^2\right)^{m_3''}R^{m_3''}\overline{D^{m_3'}}\left(\widehat{\Omega_2^3}h\right)(w)\right|dV(w)\,dV(\xi),$$

The only difference between these two iterated integrals is that one of the factors $\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}$ that occur in the first is replaced by the factor $\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}$ in the second. Note that the ignored wedge products have now been reinstated in $\widehat{\Omega_2^3}$.

Now for the iterated integral in (8.29), we can separate it into the composition of two operators of the form treated previously. One factor is the operator

$$\begin{aligned}(8.31)\qquad &\int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{\sigma}|1-\xi\overline{z}|^{n-1}}{\triangle(\xi,z)^{m_1'+m_1''+n}}\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1''}\\ &\times\left\{\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}+\triangle(\xi,z)^{m_1'}\right\}\\ &\times\left(\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\right)^{m_2'}\left|\frac{1-|\xi|^2}{1-\xi\overline{z}}\right|^{s_1-n}\left(1-|\xi|^2\right)^{-\sigma}F(\xi)\,dV(\xi),\end{aligned}$$

and the other factor is the operator

$$\begin{aligned}(8.32)\quad F(\xi)=&\int_{\mathbb{B}_n}\frac{\left(1-|\xi|^2\right)^{\sigma}\left|1-w\overline{\xi}\right|^{n-2}\left(1-|w|^2\right)}{\triangle(w,\xi)^{m_2'+m_2''+n}}\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2''}\\ &\times\left\{\left[\left(1-|\xi|^2\right)\sqrt{\triangle(w,\xi)}\right]^{m_2'}+\triangle(w,\xi)^{m_2'}\right\}\left|\frac{1-|w|^2}{1-w\overline{\xi}}\right|^{s_2-n}\\ &\times\left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{m_3'+2}\left(1-|w|^2\right)^{-\sigma}f(w)\,dV(w),\end{aligned}$$

where $f(w)=\left(1-|w|^2\right)^{\sigma}\left|\left(1-|w|^2\right)^{m_3''}R^{m_3''}\overline{D^{m_3'}}\left(\widehat{\Omega_2^3}h\right)(w)\right|$. We now show how Lemma 10 applies to obtain the appropriate boundedness.

We will in fact compare the corresponding kernels to that in (8.25). When we consider the summand $\triangle(\xi,z)^{m_1'}$ in the middle line of (8.31), the first operator has kernel

$$\begin{aligned}(8.33)\qquad &\frac{\left(1-|z|^2\right)^{\sigma+m_1''}\left(1-|\xi|^2\right)^{s_1-n-m_2'-\sigma}}{|1-\xi\overline{z}|^{s_1-2n+1}\,\triangle(\xi,z)^{m_1'+m_1''+n-\frac{m_1''+2m_1'+m_2'}{2}}}\\ &=\frac{\left(1-|z|^2\right)^{\sigma+m_1''}\left(1-|\xi|^2\right)^{s_1-3n-m_1''-\sigma}}{|1-\xi\overline{z}|^{s_1-2n+1}},\end{aligned}$$

if we choose $m_2'=m_1''+2n$ so that the factor $\triangle(\xi,z)$ disappears. This is exactly the same as the kernel of the operator in (8.25) in the previous alternative argument but with m_1'' in place of m_1' there. When we consider instead the summand $\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_1'}$ in the middle line of (8.31), we obtain the kernel in (8.33) but with $m_1''+m_1'$ in place of m_1''.

When we consider the summand $\triangle(w,\xi)^{m_2'}$ in the middle line of (8.32), the second operator has kernel

$$\begin{aligned}(8.34)\qquad & \frac{\left(1-|\xi|^2\right)^{m_2''+\sigma}\left(1-|w|^2\right)^{1+s_2-n-m_3'-2-\sigma}}{\left|1-w\overline{\xi}\right|^{s_2-2n+2}\triangle(w,\xi)^{m_2'+m_2''+n-\frac{m_2''+2m_2'+m_3'+2}{2}}} \\ = & \frac{\left(1-|\xi|^2\right)^{m_2''+\sigma}\left(1-|w|^2\right)^{s_2-3n+1-m_2''-\sigma}}{\left|1-w\overline{\xi}\right|^{s_2-2n+2}}.\end{aligned}$$

if we choose $m_3'=m_2''+2n-2$, and this is also bounded on $L^p(d\lambda_n)$ for m_2'' and s_2 sufficiently large.

Note: It is here in choosing m_2'' large that we are using the full force of Corollary 3 to perform integration by parts in the *radial* derivative m_2'' times in the first iterated integral.

When we consider instead the summand $\left[\left(1-|z|^2\right)\sqrt{\triangle(\xi,z)}\right]^{m_2'}$ in the middle line of (8.32), we obtain the kernel in (8.34) but with $m_2''+m_2'$ in place of m_2''.

To handle the iterated integral in (8.30) we must first deal with the *rogue* factor $\sqrt{\triangle(w,z)}$ whose variable pair (w,z) doesn't match that of either of the denominators $\triangle(\xi,z)$ or $\triangle(w,\xi)$. For this we use the fact that

$$\sqrt{\triangle(w,z)}=|1-w\overline{z}|\,|\varphi_z(w)|=\delta(w,z)^2\rho(w,z),$$

where $\rho(w,z)=|\varphi_z(w)|$ is the invariant pseudohyperbolic metric on the ball (Corollary 1.22 in [36]) and where $\delta(w,z)=|1-w\overline{z}|^{\frac{1}{2}}$ satisfies the triangle inequality on the ball (Proposition 5.1.2 in [24]). Thus we have

$$\begin{aligned}\rho(w,z) &\leq \rho(\xi,z)+\rho(w,\xi),\\ \delta(w,z) &\leq \delta(\xi,z)+\delta(w,\xi),\end{aligned}$$

and so also

$$\begin{aligned}\sqrt{\triangle(w,z)} &\leq 2\left[\delta(\xi,z)^2+\delta(w,\xi)^2\right]\left(|\varphi_z(\xi)|+\left|\varphi_\xi(w)\right|\right)\\ &= 2\left(1+\frac{\left|1-w\overline{\xi}\right|}{|1-\xi\overline{z}|}\right)\sqrt{\triangle(\xi,z)}+2\left(1+\frac{|1-\xi\overline{z}|}{\left|1-w\overline{\xi}\right|}\right)\sqrt{\triangle(w,\xi)}.\end{aligned}$$

Thus we can write

$$\begin{aligned}(8.35)\qquad & \frac{\sqrt{\triangle(w,z)}}{1-|w|^2}\\ \lesssim\; & \frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}+\frac{\left|1-w\overline{\xi}\right|}{1-|w|^2}\frac{1-|\xi|^2}{|1-\xi\overline{z}|}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}\\ & +\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}+\frac{|1-\xi\overline{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{\left|1-w\overline{\xi}\right|}\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}.\end{aligned}$$

All of the terms on the right hand side of (8.35) are of an appropriate form to distribute throughout the iterated integral, and again Lemma 10 applies to obtain the appropriate boundedness.

For example, the final two terms on the right side of (8.35) that involve $\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}$ are handled in the same way as the operator in (8.29) by taking $m_3' = m_2'' + 2n - 2$ and $m_2' = m_1'' + 2n$, and taking s_1 and s_2 large as required by the extra factors $\frac{|1-\xi\overline{z}|}{1-|\xi|^2}\frac{1-|\xi|^2}{|1-w\overline{\xi}|}$. With these choices the first two terms on the right side of (8.35) that involve $\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$ are then handled using Lemma 10 with $c = \pm 1$ as follows.

If we substitute the first term $\frac{1-|\xi|^2}{1-|w|^2}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$ on the right in (8.35) for the factor $\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}$ in (8.30) we get a composition of two operators as in (8.31) and (8.32) but with the kernel in (8.31) multiplied by $\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$ and the kernel in (8.32) multiplied by $\frac{1-|\xi|^2}{1-|w|^2}$ and divided by $\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}$. If we consider the summand $\triangle(\xi,z)^{m_1'}$ in the middle line of (8.31), and with the choice $m_2' = m_1'' + 2n$ already made, the first operator then has kernel

$$\begin{aligned}
&\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2} \times \frac{\left(1-|z|^2\right)^{\sigma+m_1''}\left(1-|\xi|^2\right)^{s_1-3n-m_1''-\sigma}}{|1-\xi\overline{z}|^{s_1-2n+1}} \\
= &\frac{\left(1-|z|^2\right)^{m_1''+\sigma}\left(1-|\xi|^2\right)^{s_1-m_1''-3n-1-\sigma}\sqrt{\triangle(\xi,z)}}{|1-\xi\overline{z}|^{s_1-2n+1}},
\end{aligned}$$

and hence is of the form $T_{a,b,c}$ with

$$\begin{aligned}
a &= m_1'' + \sigma, \\
b &= s_1 - 3n - 1 - m_1'' - \sigma, \\
c &= 1,
\end{aligned}$$

since $a+b+c+n+1 = s_1 - n - 1$. Now we apply Lemma 10 to conclude that this operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p\left(m_1''+\sigma\right) < -n < p\left(s_1 - 3n - m_1'' - \sigma\right),$$

i.e. $m_1'' + \sigma > \frac{n}{p}$ and $s_1 > m_1'' + \sigma + 3n - \frac{n}{p}$.

If we consider the summand $\triangle(w,\xi)^{m_2'}$ in the middle line of (8.32), and with the choice $m_3' = m_2'' + 2n - 2$ already made, the second operator has kernel

$$\begin{aligned}
&\frac{1-|\xi|^2}{1-|w|^2} \times \left(\frac{\sqrt{\triangle(w,\xi)}}{1-|w|^2}\right)^{-1} \times \frac{\left(1-|\xi|^2\right)^{m_2''+\sigma}\left(1-|w|^2\right)^{s_2-3n+1-m_2''-\sigma}}{\left|1-w\overline{\xi}\right|^{s_2-2n+2}} \\
= &\frac{\left(1-|\xi|^2\right)^{m_2''+\sigma+1}\left(1-|w|^2\right)^{s_2-3n+1-m_2''-\sigma}\sqrt{\triangle(w,\xi)}^{-1}}{\left|1-w\overline{\xi}\right|^{s_2-2n+2}},
\end{aligned}$$

and hence is of the form $T_{a,b,c}$ with

$$\begin{aligned}
a &= m_2'' + \sigma + 1, \\
b &= s_2 - 3n + 1 - m_2'' - \sigma, \\
c &= -1.
\end{aligned}$$

This operator is bounded on $L^p(\lambda_n)$ if and only if

$$-p\left(m_2''+\sigma+1\right)<-n<p\left(s_2-3n+2-m_2''-\sigma\right),$$

i.e. $m_2''+\sigma>\frac{n}{p}-1$ and $s_2>m_2''+\sigma+3n-2-\frac{n}{p}$.

If we now substitute the second term $\frac{|1-w\overline{\xi}|}{1-|w|^2}\frac{1-|\xi|^2}{|1-\xi\overline{z}|}\frac{\sqrt{\triangle(\xi,z)}}{1-|\xi|^2}$ on the right in (8.35) for the factor $\frac{\sqrt{\triangle(w,z)}}{1-|w|^2}$ in (8.30) we similarly get a composition of two operators that are each bounded on $L^p(\lambda_n)$ for m_i and s_i chosen large enough.

8.1.5. *Boundary terms for $\mathcal{F}^2$.* Now we must address in $\mathcal{F}^2$ the boundary terms that arise in the integration by parts formula (4.7). Suppose the first operator $\mathcal{C}_{n,s_1}^{0,0}$ is replaced by a boundary term, but not the second. We proceed by applying Corollary 3 to the boundary term. Since the differential operator $\left(1-|z|^2\right)^{m_1+\sigma}R^{m_1}$ hits only the kernel of the boundary term, we can apply Remark 7 to the first iterated integral and Lemma 10 to the second iterated integral in the manner indicated in the above arguments. If the second operator $\mathcal{C}_{n,s_2}^{0,1}$ is replaced by a boundary term, then as mentioned in Remark 10, the operators $\overline{D}^{m_2}$ hit only the factors $\overline{\mathcal{D}}^{m_3}$, and this produces *rogue* terms that are handled as above. If the first operator $\mathcal{C}_{n,s_1}^{0,0}$ was also replaced by a boundary term, then in addition we would have radial derivatives R^m hitting the second boundary term. Since radial derivatives are holomorphic, they hit only the holomorphic kernel and not the antiholomorphic factors in $\overline{\mathcal{D}}^{m_3}$, and so these terms can also be handled as above.

8.2. **The estimates for general $\mathcal{F}^\mu$.** In view of inequality (8.10), it suffices to establish the following inequality:

$$\begin{aligned}(8.36)\qquad &\left\|\mathcal{F}^\mu\right\|^p_{B_p^\sigma(\mathbb{B}_n)}\\ &= \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m_1+\sigma}R^{m_1}\Lambda_g\mathcal{C}_{n,s_1}^{0,0}\ldots\Lambda_g\mathcal{C}_{n,s_\mu}^{0,\mu-1}\Omega_\mu^{\mu+1}h\right|^p d\lambda_n(z)\\ &\leq C_{\sigma,n,p,\delta}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma}\mathcal{X}^{m_\mu}\left(\widehat{\Omega_\mu^{\mu+1}}h\right)(z)\right|^p d\lambda_n(z).\end{aligned}$$

Recall that the absolute value $|F|$ of an element F in the exterior algebra is the square root of the sum of the squares of the coefficients of F in the standard basis.

The case $\mu>2$ involves no new ideas, and is merely complicated by straightforward algebra. The reason is that the solution operator $\Lambda_g\mathcal{C}_{n,s_1}^{0,0}\ldots\Lambda_g\mathcal{C}_{n,s_\mu}^{0,\mu-1}$ acts *separately* in each entry of the form $\Omega_\mu^{\mu+1}h$, an element of the exterior algebra of $\mathbb{C}^\infty\otimes\mathbb{C}^n$ which we view as an alternating ℓ^2-tensor of $(0,\mu)$ forms in $\mathbb{C}^n$. These operators decompose as a sum of simpler operators with the basic property that their kernels are *identical*, except that the rogue factors in each kernel differ according to the entry. Nevertheless, there are always exactly μ distinct rogue factors in each kernel and after splitting, the μ rogue factors can be associated in one-to-one fashion with each of the $\frac{\partial}{\partial \overline{w_j}}$ derivatives in the corresponding entry of

$$\Omega_\mu^{\mu+1}h=-(\mu+1)\left(\sum_{k_0=1}^{\infty}\frac{\overline{g_{k_0}}}{|g|^2}e_{k_0}\right)\wedge\bigwedge_{i=1}^{\mu}\left(\sum_{k_i=1}^{\infty}\frac{\overline{\partial g_{k_i}}}{|g|^2}e_{k_i}\right)h.$$

After applying the crucial inequalities, this effectively results in replacing each derivative $\frac{\partial}{\partial \overline{w_j}}$ by the derivative $\overline{D_j}$, and consequently we can write the resulting form as $\widehat{\Omega_{\mu}^{\mu+1}} h$.

This completes our proof of Theorem 2.

9. Appendix

Here in the appendix we collect proofs of formulas and modifications of arguments already in the literature that would otherwise interrupt the main flow of the paper.

9.1. **Charpentier's solution kernels.** Here we prove Theorem 4. In the computation of the Cauchy kernel $\mathcal{C}_n(w,z)$, we need to compute the full exterior derivative of the section $s(w,z)$. By definition one has,

$$\begin{aligned} s_i(w,z) &= \overline{w_i}(1-w\overline{z}) - \overline{z_i}(1-|w|^2), \\ ds_i(w,z) &\equiv (\partial_w + \overline{\partial}_w + \partial_z + \overline{\partial}_z) s_i(w,z) \end{aligned}$$

Straightforward computations show that

$$\begin{aligned} \partial_w s_i(w,z) &= \sum_{j=1}^n (\overline{z}_i \overline{w}_j - \overline{w}_i \overline{z}_j)\, dw_j \\ \overline{\partial}_w s_i(w,z) &= (1-w\overline{z})\, d\overline{w}_i + \sum_{j=1}^n w_j \overline{z}_i d\overline{w}_j \\ \overline{\partial}_z s_i(w,z) &= -\sum_{j=1}^n \overline{w}_i w_j d\overline{z}_j - \left(1-|w|^2\right) d\overline{z}_i \\ \partial_z s_i(w,z) &= 0, \end{aligned} \tag{9.1}$$

as well as

$$\begin{aligned} \overline{\partial}_w s_k &= (1-w\overline{z}) d\overline{w}_k + \overline{z}_k \overline{\partial}_w |w|^2 \\ \overline{\partial}_z s_k &= -(1-|w|^2) d\overline{z}_k - \overline{w}_k \overline{\partial}_z (w\overline{z}). \end{aligned}$$

We also have the following representations of s_k, again following by simple computation. Recall from Notation 2 that $\{1,2,...,n\} = \{i_\nu\} \cup J_\nu \cup L_\nu$ where J_ν and L_ν are increasing multi-indices of lengths $n-q-1$ and q. We will use the following with $k = i_\nu$.

$$\begin{aligned} s_k &= (\overline{w}_k - \overline{z}_k) + \sum_{l \neq k} w_l (\overline{w}_l \overline{z}_k - \overline{w}_k \overline{z}_l) \\ &= (\overline{w}_k - \overline{z}_k) + \sum_{j \in J_\nu} w_j (\overline{w}_j \overline{z}_k - \overline{w}_k \overline{z}_j) + \sum_{l \in L_\nu} w_l (\overline{w}_l \overline{z}_k - \overline{w}_k \overline{z}_l) \\ &= (\overline{w}_k - \overline{z}_k) + \overline{z}_k \sum_{j \in J_\nu} |w_j|^2 - \overline{w}_k \sum_{j \in J_\nu} w_j \overline{z}_j + \overline{z}_k \sum_{l \in L_\nu} |w_l|^2 - \overline{w}_k \sum_{l \in L_\nu} w_l \overline{z}_l. \end{aligned}$$

Remark 11. *Since $A \wedge A = 0$ for any form, we have in particular that $\overline{\partial_w} |w|^2 \wedge \overline{\partial_w} |w|^2 = 0$ and $\overline{\partial_z}(w\overline{z}) \wedge \overline{\partial_z}(w\overline{z}) = 0$.*

Using this remark we next compute $\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j$. We identify J_ν as $j_1<j_2<\cdots<j_{n-q-1}$ and define a map $\imath(j_r)=r$, namely $\imath$ says where j_r occurs in the multi-index. We will frequently abuse notation and simply write $\imath(j)$. Because $\overline{\partial}_w|w|^2\wedge\overline{\partial}_w|w|^2=0$ it is easy to conclude that we can not have any term in $\overline{\partial}_w|w|^2$ of degree greater than one when expanding the wedge product of the $\overline{\partial_w}s_j$.

$$\begin{aligned}
\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j &= \bigwedge_{j\in J_\nu}\left\{(1-w\overline{z})d\overline{w}_j+\overline{z}_j\overline{\partial}_w|w|^2\right\}\\
&= (1-w\overline{z})^{n-q-1}\bigwedge_{j\in J_\nu}d\overline{w}_j+(1-w\overline{z})^{n-q-2}\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\overline{\partial}_w|w|^2\wedge\bigwedge_{j'\in J_\nu\setminus\{j\}}d\overline{w}_{j'}\\
&= (1-w\overline{z})^{n-q-2}\\
&\quad\left(\left(1-w\overline{z}+\sum_{j\in J_\nu}w_j\overline{z}_j\right)\bigwedge_{j\in J_\nu}d\overline{w}_j+\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\sum_{k\in L_\nu\cup\{i_\nu\}}w_k d\overline{w}_k\bigwedge_{j'\in J_\nu\setminus\{j\}}d\overline{w}_{j'}\right).
\end{aligned}$$

The last line follows by direct computation using

$$\overline{\partial}_w\,|w|^2=\sum_{j\in J_\nu}w_j d\overline{w}_j+\sum_{k\in L_\nu\cup\{i_\nu\}}w_k d\overline{w}_k.$$

A similar computation yields that

$$\begin{aligned}
&\bigwedge_{l\in L_\nu}\overline{\partial}_z s_l\\
=\;& (-1)^q\bigwedge_{l\in L_\nu}\left\{(1-|w|^2)d\overline{z}_l+\overline{w}_l\overline{\partial}_z(w\overline{z})\right\}\\
=\;& (-1)^q\left((1-|w|^2)^q\bigwedge_{l\in L_\nu}d\overline{z}_l+(1-|w|^2)^{q-1}\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\overline{\partial}_z(w\overline{z})\wedge\bigwedge_{l'\in L_\nu\setminus\{l\}}d\overline{z}_{l'}\right)\\
=\;& (-1)^q(1-|w|^2)^{q-1}\\
&\left(\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{l\in L_\nu}d\overline{z}_l+\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\sum_{k\in J_\nu\cup\{i_\nu\}}w_k d\overline{z}_k\bigwedge_{l'\in L_\nu\setminus\{l\}}d\overline{z}_{l'}\right).
\end{aligned}$$

An important remark at this point is that the multi-index J_ν or L_ν can only appear in the first term of the last line above. The terms after the plus sign have multi-indices that are related to J_ν and L_ν, but differ by one element. This fact will play a role later.

Combining things, we see that

$$\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j\bigwedge_{l\in L_\nu}\overline{\partial}_z s_j=(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}\left(I_\nu+II_\nu+III_\nu+IV_\nu\right),$$

where

$$I_\nu=\left(1-w\overline{z}+\sum_{j\in J_\nu}w_j\overline{z}_j\right)\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\overline{z}_l,$$

$$II_\nu=\left(1-w\overline{z}+\sum_{j\in J_\nu}w_j\overline{z}_j\right)\bigwedge_{j\in J_\nu}d\overline{w}_j\left(\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\sum_{k\in J_\nu\cup\{i_\nu\}}w_k d\overline{z}_k\bigwedge_{l'\in L_\nu\setminus\{l\}}d\overline{z}_{l'}\right),$$

$$III_\nu = \left(\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\sum_{k\in L_\nu\cup\{i_\nu\}}w_k d\overline{w}_k\bigwedge_{j'\in J_\nu\setminus\{j\}}d\overline{w}_{j'}\right)\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{l\in L_\nu}d\overline{z}_l,$$

$$\begin{aligned}IV_\nu &= \left(\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\sum_{k\in L_\nu\cup\{i_\nu\}}w_k d\overline{w}_k\bigwedge_{j'\in J_\nu\setminus\{j\}}d\overline{w}_{j'}\right)\\ &\quad\times\left(\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\sum_{k\in J_\nu\cup\{i_\nu\}}w_k d\overline{z}_k\bigwedge_{l'\in L_\nu\setminus\{l\}}d\overline{z}_{l'}\right).\end{aligned}$$

We next introduce a little more notation to aid in the computation of the kernel $\mathcal{C}_n^{0,q}(w,z)$. For $1\le k\le n$ we let $P_n^q(k)=\{\nu\in P_n^q:\nu(1)=i_\nu=k\}$. This divides the set P_n^q into n classes with $\frac{(n-1)!}{(n-q-1)!q!}$ elements. At this point, with the notation introduced in Notation 2 and computations performed above, we have reduced the calculation of $\mathcal{C}_n^{0,q}(w,z)$ to

$$\begin{aligned}\mathcal{C}_n^{0,q}(w,z) &= \frac{1}{\triangle(w,z)^n}\sum_{\nu\in P_n^q}\epsilon_\nu s_{i_\nu}\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j\bigwedge_{l\in L_\nu}\overline{\partial}_z s_l\wedge\omega(w)\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_k\sum_{\nu\in P_n^q(k)}\epsilon_\nu(I_\nu+II_\nu+III_\nu+IV_\nu)\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_k(I(k)+II(k)+III(k)+IV(k))\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_kC(k).\end{aligned}$$

Here we have defined $C(k)\equiv I(k)+II(k)+III(k)+IV(k)$, and

$$\begin{aligned}I(k)&\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu I_\nu & II(k)&\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu II_\nu\\ III(k)&\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu III_\nu & IV(k)&\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu IV_\nu.\end{aligned}$$

For a fixed $\tau\in P_n^q$ we will compute the coefficient of $\bigwedge_{j\in J_\tau}d\overline{w}_j\bigwedge_{l\in L_\tau}d\overline{z}_l$. We will ignore the functional coefficient in front of the sum since it only needs to be taken into consideration at the final stage. We will show that for this fixed τ the sum on k of s_k times $I(k)$, $II(k)$, $III(k)$ and $IV(k)$ can be replaced by $\epsilon_\tau(1-w\overline{z})(1-|w|^2)(\overline{w}_{i_\tau}-\overline{z}_{i_\tau})\bigwedge_{j\in J_\tau}d\overline{w}_j\bigwedge_{l\in L_\tau}d\overline{z}_l$. There will also be other terms that appear in this expression that arise from multi-indices J and I that are not disjoint. Using the computations below it can be seen that these terms actually vanish and hence provide no contribution for $\mathcal{C}_n^{0,q}(w,z)$. Since τ is an arbitrary element of P_n^q this will then complete the computation of the kernel.

Note that when $k = i_\tau$ then we have the following contributions. It is easy to see that $II(i_\tau) = III(i_\tau) = 0$. It is also easy to see that

$$\begin{aligned} I(i_\tau) &= \epsilon_\tau \left(1 - w\overline{z} + \sum_{j\in J_\tau} w_j\overline{z}_j\right)\left(1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l \\ &= \epsilon_\tau(1 - w\overline{z})(1 - |w|^2) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l \\ &\quad + \left((1 - w\overline{z})\sum_{l\in L_\tau} |w_l|^2 + (1 - |w|^2)\sum_{j\in J_\tau} w_j\overline{z}_j + \sum_{l\in L_\tau} |w_l|^2 \sum_{j\in J_\tau} w_j\overline{z}_j\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l. \end{aligned}$$

We also receive a contribution from term $IV(i_\tau)$ is this case. This happens by interchanging an index in the multi-index J_τ with one in L_τ. Namely, we consider the permutations $\nu : \{1, \dots, n\} \to \{i_\tau, (J_\tau \setminus \{j\}) \cup \{l\}, (L_\tau \setminus \{l\}) \cup \{j\}\}$. This permutation contributes the term $\overline{z}_l w_l \overline{w}_j w_j$. After summing over all these possible permutations, we arrive at the simplified formula,

$$IV(i_\tau) = -\epsilon_\tau \left(\sum_{j\in J_\tau} |w_j|^2\right)\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l.$$

Collecting all these terms, when $k = i_\tau$ we have that the coefficient of $\epsilon_\tau \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ is:

$$\begin{aligned} C(i_\tau) &= (1 - w\overline{z})(1 - |w|^2) + (1 - w\overline{z} + \sum_{j\in J_\tau} w_j\overline{z}_j) \sum_{l\in L_\tau} |w_l|^2 \\ &\quad + (1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2) \sum_{j\in J_\tau} w_j\overline{z}_j - \sum_{l\in L_\tau} |w_l|^2 \sum_{j\in J_\tau} w_j\overline{z}_j - \sum_{j\in J_\tau} |w_j|^2 \sum_{l\in L_\tau} w_l\overline{z}_l. \end{aligned}$$

We next note that when $k \neq i_\tau$ it is still possible to have terms which contribute to the coefficient of $\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$. To see this we further split the conditions on k into the situations where $k \in J_\tau$ and $k \in L_\tau$. First, observe in this situation that if $k \neq i_\tau$ then term $I(k)$ can never contribute. So all contributions must come from terms $II(k)$, $III(k)$, and $IV(k)$. In these terms it is possible to obtain the term $\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ by replacing some index in ν. Namely, it is possible to have ν and τ differ by one index from each other, or one by replacing an index with i_τ.

Next, observe that when $k \in L_\tau$ there exists a unique $\nu \in P_n^q(k)$ such that $J_\nu = J_\tau$. Namely, we have that $\nu : \{1, \dots, n\} \to \{k, J_\tau, (L_\tau \setminus \{k\}) \cup i_\tau\}$. Here, we used that $i_\nu = k$. Terms of this type will contribute to term $II(k)$ but will give no contribution to term $III(k)$. However, they will give a contribution to term $IV(k)$.

Similarly, when $k \in J_\tau$ there will exist a unique $\mu \in P_n^q(k)$ with $L_\mu = L_\tau$. This happens with $\mu : \{1, \dots, n\} \to \{k, (J_\tau \setminus \{k\}) \cup i_\tau, L_\tau\}$. Here we used that $i_\mu = k$. Again, we get a contribution to term $III(k)$ and $IV(k)$ and they give no contribution to the term $II(k)$.

Using these observations when $k \in L_\tau$ we arrive at the following for $I(k)$, $II(k)$, $III(k)$, and $IV(k)$:

$$\begin{aligned}
I(k) &= 0 \\
II(k) &= -\epsilon_\tau \left(1 - w\overline{z} + \sum_{j\in J_\tau} w_j \overline{z}_j\right) \overline{w}_{i_\tau} w_k \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l \\
III(k) &= 0 \\
IV(k) &= \epsilon_\tau \overline{z}_{i_\tau} w_k \left(\sum_{j\in J_\tau} |w_j|^2\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l.
\end{aligned}$$

Similarly, when $k \in J_\tau$ we arrive at the following for $I(k)$, $II(k)$, $III(k)$, and $IV(k)$:

$$\begin{aligned}
I(k) &= 0 \\
II(k) &= 0 \\
III(k) &= -\epsilon_\tau \left(1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2\right) \overline{z}_{i_\tau} w_k \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l \\
IV(k) &= \epsilon_\tau \overline{w}_{i_\tau} w_k \left(\sum_{l\in L_\tau} w_l \overline{z}_l\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l.
\end{aligned}$$

Collecting these terms, we see the following for the coefficient of $\epsilon_\tau \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$:

$$C(k) = -w_k \left(\overline{z}_{i_\tau} \left(1 - |w|^2 + \textstyle\sum_{l\in L_\tau} |w_l|^2\right) - \overline{w}_{i_\tau} \left(\textstyle\sum_{l\in L_\tau} w_l \overline{z}_l\right)\right) \qquad \forall k \in J_\tau,$$

$$C(k) = -w_k \left(\overline{w}_{i_\tau} \left(1 - w\overline{z} + \textstyle\sum_{j\in J_\tau} w_j \overline{z}_j\right) - \overline{z}_{i_\tau} \left(\textstyle\sum_{j\in J_\tau} |w_j|^2\right)\right) \qquad \forall k \in L_\tau.$$

This then implies that the *total* coefficient of $\epsilon_\tau \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ is given by

$$s_{i_\tau} C(i_\tau) + \sum_{k\in J_\tau} s_k C(k) + \sum_{k\in L_\tau} s_k C(k).$$

At this point the remainder of the proof of the Theorem 4 reduces to tedious algebra. The term $s_{i_\tau} C(i_\tau)$ will contribute the term $(1-w\overline{z})(1-|w|^2)(\overline{w}_{i_\tau} - \overline{z}_{i_\tau})$ and a remainder term. The remainder term will cancel with the terms $\sum_{k\neq i_\tau} s_k C(k)$.

We first compute the term $s_k C(k)$ for $k \in J_\tau$. Note that in this case, we have that

$$\begin{aligned}
C(k) &= w_k \left(\overline{w}_{i_\tau} \left(\sum_{l\in L_\tau} w_l \overline{z}_l\right) - \overline{z}_{i_\tau} \left(1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2\right)\right) \\
&= w_k \left(\overline{w}_{i_\tau} \left(\sum_{l\in L_\tau} w_l \overline{z}_l\right) - \overline{z}_{i_\tau} \left(1 - \sum_{l\in J_\tau} |w_l|^2\right)\right) + w_k \overline{z}_{i_\tau} |w_{i_\tau}|^2.
\end{aligned}$$

Multiplying this by s_k we see that

$$\begin{aligned} s_k C(k) &= (1-w\overline{z})\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1-\sum_{l\in J_\tau}|w_l|^2\right)\right)|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1-\sum_{l\in J_\tau}|w_l|^2\right)\right)w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2|w_k|^2 - (1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2 w_k\overline{z}_k. \end{aligned}$$

Upon summing in $k \in J_\tau$ we find that

$$\begin{aligned} \sum_{k\in J_\tau} s_k C(k) &= (1-w\overline{z})\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1-\sum_{j\in J_\tau}|w_j|^2\right)\right)\sum_{k\in J_\tau}|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1-\sum_{j\in J_\tau}|w_j|^2\right)\right)\sum_{k\in J_\tau} w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in J_\tau}|w_k|^2 - (1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in J_\tau} w_k\overline{z}_k. \end{aligned}$$

Performing similar computations for $k \in L_\tau$ we find,

$$\begin{aligned} \sum_{k\in L_\tau} s_k C(k) &= (1-w\overline{z})\left(\overline{z}_{i_\tau}\left(\sum_{k\in J_\tau}|w_j|^2\right) - \overline{w}_{i_\tau}\left(1-\sum_{l\in L_\tau} w_l\overline{z}_l\right)\right)\sum_{k\in L_\tau}|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{z}_{i_\tau}\left(\sum_{k\in J_\tau}|w_j|^2\right) - \overline{w}_{i_\tau}\left(1-\sum_{l\in L_\tau} w_l\overline{z}_l\right)\right)\sum_{k\in L_\tau} w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in L_\tau}|w_k|^2 - (1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in L_\tau} w_k\overline{z}_k. \end{aligned}$$

Putting this all together we find that

$$\begin{aligned} &\sum_{k\neq i_\tau} s_k C(k) \\ &= \overline{w}_{i_\tau}(1-w\overline{z})\left(\left(\sum_{k\in L_\tau} w_l\overline{z}_l\right)\left(\sum_{k\in J_\tau}|w_k|^2\right) - \left(1-\sum_{k\in L_\tau} w_k\overline{z}_k - w_{i_\tau}\overline{z}_{i_\tau}\right)\left(\sum_{k\in L_\tau}|w_k|^2\right)\right) \\ &\quad +\overline{z}_{i_\tau}(1-|w|^2)\left(\left(1-\sum_{k\in J_\tau}|w_k|^2 - |w_{i_\tau}|^2\right)\left(\sum_{k\in J_\tau} w_k\overline{z}_k\right) - \left(\sum_{k\in J_\tau}|w_j|^2\right)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right)\right) \\ &\quad -\overline{z}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in J_\tau}|w_j|^2\right) + \overline{w}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right). \end{aligned}$$

We next compute the term $s_{i_\tau}C(i_\tau)$. Using the properties of s_k we have that $s_{i_\tau}C(i_\tau)$ is

$$
\begin{aligned}
&(\overline{w}_{i_\tau}-\overline{z}_{i_\tau})\,(1-w\overline{z})(1-|w|^2)\\
&+\overline{z}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in J_\tau}|w_k|^2\right)-\overline{w}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in L_\tau}w_k\overline{z}_k\right)\\
&+\overline{w}_{i_\tau}(1-w\overline{z})\left\{(1-w\overline{z})\left(\sum_{k\in L_\tau}|w_k|^2\right)+\left(\sum_{k\in L_\tau}|w_k|^2\right)\left(\sum_{k\in J_\tau}w_k\overline{z}_k\right)\right.\\
&\qquad\qquad\left.-\left(\sum_{k\in J_\tau}|w_k|^2\right)\left(\sum_{k\in L_\tau}w_k\overline{z}_k\right)\right\}\\
&+\overline{z}_{i_\tau}(1-|w|^2)\left\{-(1-|w|^2)\left(\sum_{k\in J_\tau}w_k\overline{z}_k\right)-\left(\sum_{k\in L_\tau}|w_k|^2\right)\left(\sum_{k\in J_\tau}w_k\overline{z}_k\right)\right.\\
&\qquad\qquad\left.+\left(\sum_{k\in J_\tau}|w_k|^2\right)\left(\sum_{k\in L_\tau}w_k\overline{z}_k\right)\right\}.
\end{aligned}
$$

From this point on it is simple to see that the remainder of the term $s_{i_\tau}C(i_\tau)$ cancels with $\sum_{k\neq i_\tau}s_kC(k)$. One simply adds and subtracts a common term in parts of $\sum_{k\neq i_\tau}s_kC(k)$. The only term that remains is $(\overline{w}_{i_\tau}-\overline{z}_{i\tau})(1-w\overline{z})(1-|w|^2)$. Thus, we see that the term corresponding to τ in the sum $\mathcal{C}_n^{0,q}(w,z)$ is

$$
\epsilon_\tau\frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}(1-w\overline{z})(1-|w|^2)(\overline{w}_{i_\tau}-\overline{z}_{i_\tau})\bigwedge_{j\in J_\tau}d\overline{w}_j\bigwedge_{l\in L_\tau}d\overline{z}_l\wedge\omega(w).
$$

Since τ was arbitrary we conclude that $\mathcal{C}_n^{0,q}(w,z)$ equals

$$
\frac{(1-w\overline{z})^{n-q-1}\left(1-|w|^2\right)^q}{\triangle\,(w,z)^n}
$$

times

$$
\sum_{\nu\in P_n^q}\epsilon_\nu(\overline{w}_{i_\nu}-\overline{z}_{i_\nu})\bigwedge_{j\in J_\nu}d\overline{w}_j\bigwedge_{l\in L_\nu}d\overline{z}_l\wedge\omega(w),
$$

which completes the proof of Theorem 4.

9.1.1. *Explicit formulas for kernels in $n=2$ and 3 dimensions* . Using the above computations and simplifying algebra we obtain the formulas

$$
\begin{aligned}
&\mathcal{C}_2^{0,0}(w,z)\\
&=\frac{(1-w\overline{z})}{\triangle(w,z)^2}\left[(\overline{z}_2-\overline{w}_2)d\overline{w}_1\wedge dw_1\wedge dw_2-(\overline{z}_1-\overline{w}_1)d\overline{w}_2\wedge dw_1\wedge dw_2\right],
\end{aligned}
\tag{9.2}
$$

and

$$
\begin{aligned}
&\mathcal{C}_2^{0,1}(w,z)\\
&=\frac{(1-|w|^2)}{\triangle(w,z)^2}\left[(\overline{w}_2-\overline{z}_2)d\overline{z}_1\wedge dw_1\wedge dw_2-(\overline{w}_1-\overline{z}_1)d\overline{z}_2\wedge dw_1\wedge dw_2\right],
\end{aligned}
\tag{9.3}
$$

and

$$
\begin{aligned}
(9.4)\quad & \mathcal{C}_3^{0,q}(w,z) \\
= \quad & \sum_{\sigma\in\mathcal{S}_3} sgn(\sigma) \frac{(1-w\overline{z})^{2-q}\left(1-|w|^2\right)^q\left(\overline{z_{\sigma(1)}}-\overline{w_{\sigma(1)}}\right)}{\triangle(w,z)^3} d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}}\wedge\omega_3(w),
\end{aligned}
$$

where $\mathcal{S}_3$ denotes the group of permutations on $\{1,2,3\}$ and q determines the number of $d\overline{z_i}$ in the form $d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}}$:

$$
d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}} = \begin{cases} d\overline{w_{\sigma(2)}}\wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q=0 \\ d\overline{z_{\sigma(2)}}\wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q=1 \\ d\overline{z_{\sigma(2)}}\wedge d\overline{z_{\sigma(3)}} & \text{if} \quad q=2 \end{cases}.
$$

9.1.2. *Integrating in higher dimensions.* Here we give the proof of Lemma 1. Let

$$
B \equiv \frac{\left(1-|z|^2\right)}{|1-w\overline{z}|^2} \text{ and } R \equiv \sqrt{1-|w|^2},
$$

so that

$$
BR^2 = \frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2} = 1-|\varphi_w(z)|^2 .
$$

Then with the change of variable $\rho = Br^2$ we obtain

$$
\begin{aligned}
& (1-w\overline{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_k} \frac{(1-|w|^2-|w'|^2)^q}{\triangle((w,w'),(z,0))^s} dV(w') \\
= \quad & \frac{(1-w\overline{z})^{s-q-1}}{|1-w\overline{z}|^{2s}} \int_{\sqrt{1-|w|^2}\mathbb{B}_k} \frac{\left(1-|w|^2-|w'|^2\right)^q}{\left(1-\frac{(1-|z|^2)}{|1-w\overline{z}|^2}\left(1-|w|^2-|w'|^2\right)\right)^s} dV(w') \\
= \quad & \frac{(1-w\overline{z})^{s-q-1}}{|1-w\overline{z}|^{2s}} \int_0^R \frac{\left(R^2-r^2\right)^q}{\left(1-BR^2+Br^2\right)^s} r^{2k-1} dr \\
= \quad & \frac{(1-w\overline{z})^{s-q-1}}{2B^{k+q}|1-w\overline{z}|^{2s}} \int_0^{BR^2} \frac{\left(BR^2-\rho\right)^q}{\left(1-BR^2+\rho\right)^s}\rho^{k-1} d\rho,
\end{aligned}
$$

which with

$$
\Psi_{n,k}^{0,q}(t) = \frac{(1-t)^n}{t^k}\int_0^t \frac{(t-\rho)^q}{(1-t+\rho)^{n+k}}\rho^{k-1} d\rho,
$$

we rewrite as

$$\begin{aligned}
&\frac{(1-w\overline{z})^{s-q-1}}{2B^{k+q}\left|1-w\overline{z}\right|^{2s}}\frac{(BR^2)^k}{\left|\varphi_w(z)\right|^{2n}}\Psi_{n,k}^{0,q}\left(BR^2\right)\\
&= \frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^k}{2\left(1-|z|^2\right)^q\left|1-w\overline{z}\right|^{2s}}\frac{\left|1-w\overline{z}\right|^{2q}}{\left|\varphi_w(z)\right|^{2n}}\Psi_{n,k}^{0,q}\left(BR^2\right)\\
&= \frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^k}{2\left(1-|z|^2\right)^q}\frac{\left|1-w\overline{z}\right|^{2q-2k}}{\triangle(w,z)^n}\Psi_{n,k}^{0,q}\left(BR^2\right)\\
&= \frac{1}{2}\Phi_n^q(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{k-q}\left(\frac{1-w\overline{z}}{1-|z|^2}\right)^q\Psi_{n,k}^{0,q}\left(BR^2\right).
\end{aligned}$$

since $\Phi_n^q(w,z)=\frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}$.

At this point we claim that

$$\Psi_{n,k}^{0,q}(t)=\frac{(1-t)^n}{t^k}\int_0^t\frac{(t-r)^q}{(1-t+r)^{n+k}}r^{k-1}dr \tag{9.5}$$

is a polynomial in

$$t=BR^2=1-\left|\varphi_w(z)\right|^2$$

of degree $n-1$ that vanishes to order q at $t=0$, so that

$$\Psi_{n,k}^{0,q}(t)=\sum_{j=q}^{n-1}c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{\left|1-w\overline{z}\right|^2}\right)^j,$$

With this claim established, the proof of Lemma 1 is complete.

To see that $\Psi_{n,k}^{0,q}$ vanishes of order q at $t=0$ is easy since for t small (9.5) yields

$$\left|\Psi_{n,k}^{0,q}(t)\right|\le Ct^{-k}\int_0^t\frac{t^q}{C}r^{k-1}dr\le Ct^q.$$

To see that $\Psi_{n,k}^{0,q}$ is a polynomial of degree $n-1$ we prove two recursion formulas valid for $0\le t<1$ (we let $t\to 1$ at the end of the argument):

$$\begin{aligned}
\Psi_{n,k}^{0,q}(t)-\Psi_{n,k}^{0,q+1}(t) &= (1-t)\Psi_{n-1,k}^{0,q}(t),\\
\Psi_{n,k}^{0,0}(t) &= \frac{1}{k}(1-t)^n+\frac{n+k}{k}t\Psi_{n,k+1}^{0,0}(t).
\end{aligned} \tag{9.6}$$

The first formula follows from

$$(t-r)^q-(t-r)^{q+1}=(t-r)^q(1-t+r),$$

while the second is an integration by parts:

$$\begin{aligned}
\int_0^t\frac{r^{k-1}}{(1-t+r)^{n+k}}dr &= \frac{1}{k}\frac{r^k}{(1-t+r)^{n+k}}\Big|_0^t\\
&\quad+\frac{n+k}{k}\int_0^t\frac{r^k}{(1-t+r)^{n+k+1}}dr\\
&= \frac{1}{k}t^k+\frac{n+k}{k}\int_0^t\frac{r^k}{(1-t+r)^{n+k+1}}dr.
\end{aligned}$$

If we multiply this equality through by $\frac{(1-t)^n}{t^k}$ we obtain the second formula in (9.6).

The recursion formulas in (9.6) reduce matters to proving that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$. Indeed, once we know that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$, then the second formula in (9.6) and induction on k shows that $\Psi_{n,k}^{0,0}$ is as well. Then the first formula and induction on q then shows that $\Psi_{n,k}^{0,q}$ is also. To see that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$ we compute

$$\begin{aligned}
\Psi_{n,1}^{0,0}(t) &= \frac{(1-t)^n}{t}\int_0^t \frac{1}{(1-t+r)^{n+1}}dr \\
&= \frac{(1-t)^n}{t}\left\{-\frac{1}{n(1-t+r)^n}\right\}\Big|_0^t \\
&= \frac{1-(1-t)^n}{nt},
\end{aligned}$$

which is a polynomial of degree $n-1$. This finishes the proof of the claim, and hence that of Lemma 1 as well.

9.2. **Integration by parts formulas in the ball.** We begin by proving the generalized analogue of the integration by parts formula of Ortega and Fabrega [20] as given in Lemma 3. For this we will use the following identities.

Lemma 11. *For $\ell \in \mathbb{Z}$, we have*

$$\begin{aligned}
\overline{\mathcal{Z}}\left\{\triangle(w,z)^\ell\right\} &= \ell\,\triangle(w,z)^\ell, \\
\overline{\mathcal{Z}}\left\{(1-w\overline{z})^\ell\right\} &= 0, \\
\overline{\mathcal{Z}}\left\{\left(1-|w|^2\right)^\ell\right\} &= \ell\left(1-|w|^2\right)^\ell-\ell\left(1-|w|^2\right)^{\ell-1}(1-\overline{z}w).
\end{aligned}\tag{9.7}$$

Proof: (of Lemma 11) The computation

$$\begin{aligned}
\frac{\partial\triangle}{\partial\overline{w_j}} &= \frac{\partial}{\partial\overline{w_j}}\left\{|1-w\overline{z}|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)\right\} \\
&= (w\overline{z}-1)\,z_j+\left(1-|z|^2\right)w_j,
\end{aligned}$$

shows that $\overline{\mathcal{Z}}\triangle = \triangle$:

$$\begin{aligned}
\overline{\mathcal{Z}}\,\triangle(w,z) &= \left(\sum_{j=1}^n(\overline{w_j}-\overline{z_j})\frac{\partial}{\partial\overline{w_j}}\right)\left\{|1-w\overline{z}|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)\right\} \\
&= \sum_{j=1}^n(\overline{w_j}-\overline{z_j})\left\{(w\overline{z}-1)\,z_j+\left(1-|z|^2\right)w_j\right\} \\
&= \left(\overline{w}z-|z|^2\right)(w\overline{z}-1)+\left(1-|z|^2\right)\left(|w|^2-\overline{z}w\right) \\
&= -\overline{w\overline{z}}+|z|^2+|\overline{w}z|^2-|z|^2\,w\overline{z}+|w|^2-w\overline{z}-|z|^2\,|w|^2+|z|^2\,w\overline{z} \\
&= -2\operatorname{Re}w\overline{z}+|z|^2+|w\overline{z}|^2+|w|^2-|z|^2\,|w|^2 \\
&= |w-z|^2+|w\overline{z}|^2-|z|^2\,|w|^2=\triangle(w,z)
\end{aligned}$$

by the second line in (3.1) above. Iteration then gives the first line in (9.7). The second line is trivial since $1-w\overline{z}$ is holomorphic in w. The third line follows by iterating

$$\overline{\mathcal{Z}}\left(1-|w|^2\right)=\overline{z}w-|w|^2=\left(1-|w|^2\right)-(1-\overline{z}w).$$

Proof of Lemma 3: We use the general formula (3.10) for the solution kernels $\mathcal{C}_n^{0,q}$ to prove (4.7) by induction on m. For $m=0$ we obtain

(9.8)
$$\mathcal{C}_n^{0,q}\eta(z)=c_0\int_{\mathbb{B}_n}\Phi_n^q(w,z)\left\{\sum_{|J|=q}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\overline{z}^J\right\}dV(w)\equiv c_0\Phi_n^q\left(\overline{\mathcal{D}^0}\eta\right)(z),$$

from (4.5) and the following calculation using (3.9):

$$\begin{aligned}
&\mathcal{C}_n^{0,q}\eta(z)\\
\equiv\ &\int_{\mathbb{B}_n}\mathcal{C}_n^{0,q}(w,z)\wedge\eta(w)\\
=\ &\int_{\mathbb{B}_n}\sum_{|J|=q}\Phi_n^q(w,z)\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\overline{z_k}-\overline{\eta_k}\right)d\overline{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w)\wedge\left(\sum_{|I|=q+1}\eta_I d\overline{w_I}\right)\\
=\ &\left\{\int_{\mathbb{B}_n}\Phi_n^q(w,z)\left[\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\overline{z_k}-\overline{w_k}\right)\eta_{J\cup\{k\}}d\overline{z}^J\right]dV(w)\right\}.
\end{aligned}$$

Now we consider the case $m=1$. First we note that for each J with $|J|=q$,

$$\overline{\mathcal{Z}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)-\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)=\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right).\tag{9.9}$$

Indeed, we compute

$$\begin{aligned}
\overline{\mathcal{Z}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)&=\left(\sum_{j=1}^n\left(\overline{w_j}-\overline{z_j}\right)\frac{\partial}{\partial\overline{w_j}}\right)\left(\sum_{k\notin J}\left(\overline{w_k}-\overline{z_k}\right)\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\eta_I\right)\\
&=\sum_{j=1}^n\sum_{k\notin J}\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\left(\overline{w_j}-\overline{z_j}\right)\left(\overline{w_k}-\overline{z_k}\right)\frac{\partial}{\partial\overline{w_j}}\eta_I\\
&\quad+\sum_{k\notin J}\left(\overline{w_k}-\overline{z_k}\right)\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\eta_I,
\end{aligned}$$

so that

$$\begin{aligned}
&\overline{\mathcal{Z}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)-\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\\
=\ &\sum_{j=1}^n\sum_{k\notin J}\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\left(\overline{w_j}-\overline{z_j}\right)\left(\overline{w_k}-\overline{z_k}\right)\frac{\partial}{\partial\overline{w_j}}\eta_I=\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right).
\end{aligned}$$

For $|J|=q$ and $0\leq\ell\leq q$ define

$$\mathcal{I}_J^\ell\equiv\sum_{j=1}^n\int_{\mathbb{B}_n}\frac{\partial}{\partial\overline{w_j}}\left\{\frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(w,z)^n}\left(\overline{w_j}-\overline{z_j}\right)\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\right\}\omega(\overline{w})\wedge\omega(w).$$

By (3) and (4) of Proposition 16.4.4 in [24] we have

$$\sum_{j=1}^{n}(-1)^{j-1}\left(\overline{w_j}-\overline{z_j}\right)\bigwedge_{k\neq j}d\overline{w_k}\wedge\omega(w)\mid_{\partial\mathbb{B}_n}=c\left(1-\overline{z}w\right)d\sigma(w),$$

and Stokes' theorem then yields

$$\mathcal{I}_J^{\ell}=c\int_{\partial\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle(w,z)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\sigma(w)=0,$$

since $\ell\geq 1$ and $1-|w|^2$ vanishes on $\partial\mathbb{B}_n$. Moreover, from Lemma 11 we obtain

$$\begin{aligned}
\mathcal{I}_J^{\ell} &= n\int_{\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-1-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\\
&\quad+\int_{\mathbb{B}_n}\overline{\mathcal{Z}}\left\{\frac{\left(1-w\overline{z}\right)^{n-1-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\right\}dV(w)\\
&= \int_{\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-1-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle(z,w)^n}\overline{\mathcal{Z}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\\
&\quad+\ell\int_{\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-1-\ell}\left(1-|w|^2\right)^{\ell}}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\\
&\quad-\ell\int_{\mathbb{B}_n}\frac{\left(1-w\overline{z}\right)^{n-\ell}\left(1-|w|^2\right)^{\ell-1}}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w).
\end{aligned}$$

Combining this with (9.9) and (9.8) yields

$$\begin{aligned}
\Phi_n^{\ell}\left(\overline{\mathcal{D}^0}\eta\right)(z) &= \sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell}(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&= \sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell}(w,z)\,\overline{\mathcal{Z}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&\quad-\sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell}(w,z)\,\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&= -\sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell}(w,z)\,\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&\quad-\ell\sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell}(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&\quad+\ell\sum_J\int_{\mathbb{B}_n}\Phi_n^{\ell-1}(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J\\
&= -\Phi_n^{\ell}\left(\overline{\mathcal{D}^1}\eta\right)(z)-\ell\Phi_n^{\ell}\left(\overline{\mathcal{D}^0}\eta\right)(z)+\ell\Phi_n^{\ell-1}\left(\overline{\mathcal{D}^0}\eta\right)(z).
\end{aligned}$$

Thus we have

$$\Phi_n^{\ell}\left(\overline{\mathcal{D}^0}\eta\right)(z)=-\frac{1}{\ell+1}\Phi_n^{\ell}\left(\overline{\mathcal{D}^1}\eta\right)(z)+\frac{\ell}{\ell+1}\Phi_n^{\ell-1}\left(\overline{\mathcal{D}^0}\eta\right)(z).\tag{9.10}$$

From (9.8) and then iterating (9.10) we obtain

$$
\begin{aligned}
(9.11)\quad & \mathcal{C}_n^{(0,q)}\eta(z) \\
&= \Phi_n^q\left(\overline{\mathcal{D}^0}\eta\right)(z) = -\frac{1}{q+1}\Phi_n^q\left(\overline{\mathcal{D}^1}\eta\right)(z) + \frac{q}{q+1}\Phi_n^{q-1}\left(\overline{\mathcal{D}^0}\eta\right)(z) \\
&= -\frac{1}{q+1}\Phi_n^q\left(\overline{\mathcal{D}^1}\eta\right)(z) + \frac{q}{q+1}\left\{-\frac{1}{q}\Phi_n^{q-1}\left(\overline{\mathcal{D}^1}\eta\right)(z) + \frac{q-1}{q}\Phi_n^{q-2}\left(\overline{\mathcal{D}^0}\eta\right)(z)\right\} \\
&= -\frac{1}{q+1}\sum_{\ell=1}^{q}\Phi_n^\ell\left(\overline{\mathcal{D}^1}\eta\right)(z) + \textit{boundary term}.
\end{aligned}
$$

Thus we have obtained the second sum in (4.7) with $c_\ell = -\frac{1}{q+1}$ for $1 \le \ell \le q$ in the case $m = 1$.

We have included *boundary term* in (9.11) since when we use Stokes' theorem on $\Phi_n^0\left(\overline{\mathcal{D}^0}\eta\right)$ the boundary integral no longer vanishes. In fact when $\ell = 0$ the boundary term in Stokes' theorem is

$$
\begin{aligned}
\mathcal{I}_J^0 &= c\int_{\partial\mathbb{B}_n} \frac{(1-\zeta\overline{z})^n}{\triangle(\zeta,z)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right) d\sigma(\zeta) \\
&= c\int_{\partial\mathbb{B}_n} \frac{1}{\left(1-\overline{\zeta}z\right)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right) d\sigma(\zeta),
\end{aligned}
$$

since from (3.4) we have

$$
\frac{(1-w\overline{z})^n}{\triangle(z,w)^n} = \frac{(1-w\overline{z})^n}{|1-w\overline{z}|^{2n}\,|\varphi_z(w)|^{2n}} = \frac{1}{(1-\overline{w}z)^n}, \qquad w \in \partial\mathbb{B}_n.
$$

Thus the boundary term in (9.11) is

$$
c\sum_J \int_{\partial\mathbb{B}_n} \frac{1}{\left(1-\overline{\zeta}z\right)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right) d\sigma(\zeta)\, d\overline{z}^J = c\mathcal{S}_n\left(\overline{\mathcal{D}^0}\eta\right)(z).
$$

This completes the proof of (4.7) in the case $m = 1$. Now we proceed by induction on m to complete the proof of Lemma 3.

Finally here is the simple proof of the integration by parts formula for the radial derivative in Lemma 4.

Proof of Lemma 4: Since $\left(1-|w|^2\right)^{b+1}$ vanishes on the boundary for $b > -1$, and since

$$
R\left(1-|w|^2\right)^{b+1} = \sum_{j=1}^n w_j \frac{\partial}{\partial w_j}\left(1-|w|^2\right)^{b+1} = -(b+1)\left(1-|w|^2\right)^b |w|^2,
$$

the divergence theorem yields

$$\begin{aligned} 0 &= \int_{\partial\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, w\cdot \mathbf{n} d\sigma(w) \\ &= \int_{\mathbb{B}_n} \sum_{j=1}^n \frac{\partial}{\partial w_j} \left\{ w_j \left(1-|w|^2\right)^{b+1} \Psi(w) \right\} dV(w) \\ &= n\int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, dV(w) \\ &\quad + (b+1)\int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b} \left(-|w|^2\right) \Psi(w)\, dV(w) \\ &\quad + \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} R\Psi(w)\, dV(w), \end{aligned}$$

which after rearranging becomes

$$\begin{aligned} & (n+b+1)\int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, dV(w) \\ & + \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} R\Psi(w)\, dV(w). \\ = \; & (b+1)\int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b} \Psi(w)\, dV(w). \end{aligned}$$

9.3. **Equivalent seminorms on Besov-Sobolev spaces.** It is a routine matter to take known scalar-valued proofs of the results in this section and replace the scalars with vectors in ℓ^2 to obtain proofs for the ℓ^2-valued versions. We begin illustrating this process by proving the equivalence of norms in Proposition 1.

Proof of Proposition 1: First we note the equivalence of the following two conditions (the case $\sigma = 0$ is Theorem 6.1 of [36]):

(1) The functions

$$\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z), \qquad |k| = N$$

are in $L^p\left(d\lambda_n; \ell^2\right)$ for some $N > \frac{n}{p} - \sigma$,

(2) The functions

$$\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z), \qquad |k| = N$$

are in $L^p\left(d\lambda_n; \ell^2\right)$ for every $N > \frac{n}{p} - \sigma$.

Indeed, $L^p\left(d\lambda_n; \ell^2\right) = L^p\left(\nu_{-n-1}; \ell^2\right)$ and $\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z) \in L^p\left(\nu_{-n-1}; \ell^2\right)$ if and only if $\frac{\partial^{|k|}}{\partial z^k} f(z) \in L^p\left(\nu_{p(|k|+\sigma)-n-1}; \ell^2\right)$. Provided $p(|k|+\sigma) - n - 1 > -1$, Theorem 2.17 of [36] shows that $\left(1-|z|^2\right)^{\ell} \frac{\partial^{|\ell|}}{\partial z^\ell} \left(\frac{\partial^{|k|}}{\partial z^k} f\right)(z) \in L^p\left(\nu_{p(|k|+\sigma)-n-1}; \ell^2\right)$, which shows that (2) follows from (1).

From the equivalence of (1) and (2) we obtain the equivalence of the first two conditions in Proposition 1. The equivalence with the next two conditions follows from the corresponding generalization to $\sigma > 0$ of Theorem 6.4 in [36], which in turn is achieved by arguing as in the previous paragraph.

Next we prove Lemma 7 by adapting the proof of Lemma 6.4 in [6].

Proof of Lemma 7: We have

$$|D_a f(z)| = \left| f'(z) \left\{ \left(1-|a|^2\right) P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a \right\} \right| \geq \left| \left(1-|a|^2\right) f'(z) \right|, \tag{9.12}$$

and iterating with f replaced by (the components of) $D_a f$ in (9.12), we obtain

$$\left|D_a^2 f(z)\right| \geq \left| \left(1-|a|^2\right) (D_a f)'(z) \right|.$$

Applying (9.12) once more with f replaced by (the components of) f', we get

$$\left| \left(1-|a|^2\right) (D_a f)'(z) \right| = \left| \left(1-|a|^2\right) D_a (f')(z) \right| \geq \left| \left(1-|a|^2\right)^2 f''(z) \right|,$$

which when combined with the previous inequality yields

$$\left|D_a^2 f(z)\right| \geq \left| \left(1-|a|^2\right)^2 f''(z) \right|.$$

Continuing by induction we have

$$|D_a^m f(z)| \geq \left| \left(1-|a|^2\right)^m f^{(m)}(z) \right|, \qquad m \geq 1. \tag{9.13}$$

Proposition 1 and (9.13) now show that

$$\begin{aligned}
& \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m+\sigma} R^{0,m} f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} \\
\leq \;& C \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \\
\leq \;& C \left(\sum_{\alpha\in\mathcal{T}_n} \int_{B_\beta(c_\alpha,C_2)} \left| \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \\
\leq \;& C \left(\sum_{\alpha\in\mathcal{T}_n} \int_{B_\beta(c_\alpha,C_2)} \left| \left(1-|c_\alpha|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \\
\leq \;& C \left(\sum_{\alpha\in\mathcal{T}_n} \int_{B_\beta(c_\alpha,C_2)} \left| \left(1-|z|^2\right)^{\sigma} D_{c_\alpha}^m f(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \\
= \;& C \|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right|.
\end{aligned}$$

For the opposite inequality, just as in [6], we employ some of the ideas in the proofs of Theorem 6.11 and Lemma 3.3 in [36], where the case $\sigma = 0$ and $m = 1 > \frac{2n}{p}$ is proved. Suppose $f \in H(\mathbb{B}_n)$ and that the right side of (6.5) is finite. By Proposition 1 and the proof of Theorem 6.7 of [36] we have

$$f(z) = c \int_{\mathbb{B}_n} \frac{g(w)}{(1-\overline{w}z)^{n+1+\sigma}} dV(w), \qquad z \in \mathbb{B}_n, \tag{9.14}$$

for some $g \in L^p(\lambda_n)$ where

$$\|g\|_{L^p(\lambda_n)} \approx \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| + \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \tag{9.15}$$

Indeed, Proposition 1 shows that

$$\begin{aligned} f &\in B_p^\sigma(\mathbb{B}_n) \Leftrightarrow \left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z) \in L^p(\lambda_n) \\ &\Leftrightarrow R^{\sigma,m} f(z) \in L^p\left(\nu_{p(m+\sigma)-n-1}\right) \cap H(\mathbb{B}_n), \end{aligned}$$

where as in [36] we write $d\nu_\alpha(z) = \left(1-|z|^2\right)^\alpha dV(z)$. Now Lemma 10 above (see also Proposition 2.11 in [36]) shows that

$$T_{0,\beta,0} L^p(\nu_\gamma) = L^p(\nu_\gamma) \cap H(\mathbb{B}_n)$$

if and only if $p(\beta+1) > \gamma+1$. Choosing $\beta = m+\sigma$ and $\gamma = p(m+\sigma)-n-1$ we see that $p(\beta+1) > \gamma+1$ and so $f \in B_p^\sigma(\mathbb{B}_n)$ if and only if

$$R^{\sigma,m} f(z) = c \int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^{m+\sigma} h(w)}{(1-\overline{w}z)^{n+1+m+\sigma}} dV(w)$$

for some $h \in L^p\left(\nu_{p(m+\sigma)-n-1}\right)$. If we set $g(w) = \left(1-|w|^2\right)^{m+\sigma} h(w)$ we obtain

$$R^{\sigma,m} f(z) = c \int_{\mathbb{B}_n} \frac{g(w)}{(1-\overline{w}z)^{n+1+m+\sigma}} dV(w) \tag{9.16}$$

with $g \in L^p(\lambda_n)$. Now apply the inverse operator $R_{\sigma,m} = (R^{\sigma,m})^{-1}$ to both sides of (9.16) and use (6.3),

$$R_{\sigma,m}\left(\frac{1}{(1-\overline{w}z)^{n+1+m+\sigma}}\right) = \frac{1}{(1-\overline{w}z)^{n+1+\sigma}},$$

to obtain (9.14) and (9.15).

Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha \in \mathbb{B}_n$. We claim that

$$|D_a^m f(z)| \le C_m \left(1-|a|^2\right)^{\frac{m}{2}} \int_{\mathbb{B}_n} \frac{|g(w)|}{|1-\overline{w}z|^{n+1+\frac{m}{2}+\sigma}} dV(w), \qquad m \ge 1, z \in B_\beta(a,C). \tag{9.17}$$

To see this we compute $D_a^m f(z)$ for $z \in B_\beta(a,C)$, beginning with the case $m=1$. Since

$$\begin{aligned} D_a(\overline{w}z) &= (\overline{w}z)' \varphi_a'(0) = -\overline{w}^t \left\{\left(1-|a|^2\right) P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\} \\ &= -\overline{\left\{\left(1-|a|^2\right) P_a w + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a w\right\}}^t, \end{aligned}$$

we have

$$(9.18)\quad D_a f(z) = c_n \int_{\mathbb{B}_n} D_a (1-\overline{w}z)^{-(n+1+\sigma)} g(w)\, dV(w)$$
$$= c_n \int_{\mathbb{B}_n} (1-\overline{w}z)^{-(n+2+\sigma)} D_a(\overline{w}z)\, g(w)\, dV(w)$$
$$= c_n \int_{\mathbb{B}_n} (1-\overline{w}z)^{-(n+2+\sigma)} \overline{\left\{\left(1-|a|^2\right) P_a w + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a w\right\}}^t g(w)\, dV(w).$$

Taking absolute values inside, we obtain

$$(9.19)\quad |D_a f(z)| \le C\left(1-|a|^2\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{\left(1-|a|^2\right)^{\frac{1}{2}} |P_a w| + |Q_a w|}{|1-\overline{w}z|^{n+2+\sigma}} |g(w)|\, dV(w).$$

From the following elementary inequalities

$$(9.20)\quad \begin{aligned} |Q_a w|^2 &= |Q_a(w-a)|^2 \le |w-a|^2, \\ &= |w|^2 + |a|^2 - 2\operatorname{Re}(w\overline{a}) \\ &\le 2\operatorname{Re}(1-w\overline{a}) \le 2|1-w\overline{a}|, \end{aligned}$$

we obtain that $|Q_a w| \le C|1-\overline{w}a|^{\frac{1}{2}}$. Now

$$|1-w\overline{a}| \approx |1-w\overline{z}| \ge \frac{1}{2}\left(1-|z|^2\right) \approx \left(1-|a|^2\right), \qquad z \in B_\beta(a,C)$$

shows that

$$\left(1-|a|^2\right)^{\frac{1}{2}} + |1-\overline{w}a|^{\frac{1}{2}} \le C|1-\overline{w}z|^{\frac{1}{2}}, \qquad z \in B_\beta(a,C),$$

and so we see that

$$\frac{\left(1-|a|^2\right)^{\frac{1}{2}} |P_a w| + |Q_a w|}{|1-\overline{w}z|^{n+2}} \le \frac{C}{|1-\overline{w}z|^{n+\frac{3}{2}}}, \qquad z \in B_\beta(a,C).$$

Plugging this estimate into (9.19) yields

$$|D_a f(z)| \le C\left(1-|a|^2\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{|g(w)|}{|1-\overline{w}z|^{n+\frac{3}{2}+\sigma}} dV(w),$$

which is the case $m=1$ of (9.17).

To obtain the case $m=2$ of (9.17), we differentiate (9.18) again to get

$$D_a^2 f(z) = c\int_{\mathbb{B}_n} (1-\overline{w}z)^{-(n+3+\sigma)} W\overline{W}^t g(w)\, dV(w).$$

where we have written $W = \left\{\left(1-|a|^2\right) P_a w + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a w\right\}$ for convenience. Again taking absolute values inside, we obtain

$$\left|D_a^2 f(z)\right| \le C\left(1-|a|^2\right) \int_{\mathbb{B}_n} \frac{\left(\left(1-|a|^2\right)^{\frac{1}{2}} |P_a w| + |Q_a w|\right)^2}{|1-\overline{w}z|^{n+3+\sigma}} |g(w)|\, dV(w).$$

Once again, using $|Q_a w| \leq C\,|1-\overline{w}a|^{\frac{1}{2}}$ and $\left(1-|a|^2\right)^{\frac{1}{2}} + |1-\overline{w}a|^{\frac{1}{2}} \leq C\,|1-\overline{w}z|^{\frac{1}{2}}$ for $z \in B_\beta(a,C)$, we see that

$$\frac{\left(\left(1-|a|^2\right)^{\frac{1}{2}}|P_a w| + |Q_a w|\right)^2}{|1-\overline{w}z|^{n+3+\sigma}} \leq \frac{C}{|1-\overline{w}z|^{n+2+\sigma}}, \qquad z \in B_\beta(a,C),$$

which yields the case $m=2$ of (9.17). The general case of (9.17) follows by induction on m.

The inequality (9.17) shows that $\left(1-|z|^2\right)^\sigma \left|D^m_{c_\alpha} f(z)\right| \leq C_m S|g|(z)$ for $z \in B_\beta(c_\alpha, C)$, where

$$Sg(z) = \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{\frac{m}{2}+\sigma}}{|1-\overline{w}z|^{n+1+\frac{m}{2}+\sigma}} g(w)\,dV(w).$$

We will now use the symbol a differently than before. The operator S is the operator $T_{a,b,c}$ in Lemma 10 above (see also Theorem 2.10 of [36]) with parameters $a = \frac{m}{2}+\sigma$ and $b=c=0$. Now with $t=-n-1$, our assumption that $m > 2\left(\frac{n}{p}-\sigma\right)$ yields $-p\left(\frac{m}{2}+\sigma\right) < -n < p(0+1)$, i.e.

$$-pa < t+1 < p(b+1).$$

Thus the bounded overlap property of the balls $B_\beta(c_\alpha, C_2)$ together with Lemma 10 above yields

$$\begin{aligned}
\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} &= \left(\sum_{\alpha\in\mathcal{T}_n} \int_{B_\beta(c_\alpha,C_2)} \left|\left(1-|z|^2\right)^\sigma D^m_{c_\alpha} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \\
&\leq C_m \left(\int_{\mathbb{B}_n} |Sg(z)|^p\, d\lambda_n(z)\right)^{\frac{1}{p}} \\
&\leq C'_m \left(\int_{\mathbb{B}_n} |g(z)|^p\, d\lambda_n(z)\right)^{\frac{1}{p}} \\
&\leq C''_m \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}
\end{aligned}$$

by (9.15). This completes the proof of Lemma 7.

9.3.1. *Multilinear inequalities.* Proposition 3 is proved by adapting the proof of Theorem 3.5 in Ortega and Fabrega [20] to ℓ^2-valued functions. This argument uses the complex interpolation theorem of Beatrous [11] and Ligocka [17], which extends to Hilbert space valued functions with the same proof. In order to apply this extension we will need the following operator norm inequality.

If $\varphi \in M_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ and $f = \sum_{|I|=\kappa} f_I e_I \in B^\sigma_p\left(\mathbb{B}_n; \otimes^{\kappa-1}\ell^2\right)$, we define

$$\mathbb{M}_\varphi f = \varphi \otimes f = \varphi \otimes \left(\sum_{|I|=\kappa-1} f_I e_I\right) = \sum_{|I|=\kappa-1} (\varphi f_I) \otimes e_I,$$

where $I = (i_1, ..., i_{\kappa-1}) \in \mathbb{N}^{\kappa-1}$ and $e_I = e_{i_1} \otimes ... \otimes e_{i_{\kappa-1}}$.

Lemma 12. *Suppose that $\sigma \geq 0$, $1 < p < \infty$ and $\kappa \geq 1$. Then there is a constant $C_{n,\sigma,p,\kappa}$ such that*

$$\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_p^\sigma(\mathbb{B}_n;\otimes^\kappa\ell^2)} \leq C_{n,\sigma,p,\kappa}\,\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n;\ell^2)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\,. \tag{9.21}$$

In the case $p=2$ we have equality:

$$\|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)\to B_2^\sigma(\mathbb{B}_n;\otimes^\kappa\ell^2)} = \|\mathbb{M}_\varphi\|_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}\,. \tag{9.22}$$

It turns out that in order to prove (9.21) for $p \neq 2$ we will need the case $M=1$ of Proposition 3. Fortunately, the case $M=1$ does not require inequality (9.21), thus avoiding circularity.

Proof of Proposition 3 and Lemma 12: We begin with the proof of the case $M=1$ of Proposition 3. We will show that for $m=\ell+k$,

$$\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right)\right|^p d\lambda_n(z) \leq C_{n,\sigma,p}\,\|\mathbb{M}_g\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\,\|h\|^p_{B_p^\sigma(\mathbb{B}_n)}\,. \tag{9.23}$$

Following the proof of Theorem 3.1 in [20] we first convert the Leibniz formula

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \mathcal{Y}^\ell\left(g\mathcal{Y}^k h\right) - \sum_{\alpha=0}^{\ell-1}\begin{pmatrix}\ell\\ \alpha\end{pmatrix}\left(\mathcal{Y}^\alpha g\right)\left(\mathcal{Y}^{k+\ell-\alpha}h\right)$$

to "divergence form"

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \sum_{\alpha=0}^{\ell}(-1)^\alpha\begin{pmatrix}\ell\\ \ell-\alpha\end{pmatrix}\mathcal{Y}^{\ell-\alpha}\left(g\mathcal{Y}^{k+\alpha}h\right).$$

This is easily established by induction on ℓ with k held fixed and can be stated as

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \sum_{\alpha=0}^{\ell} c_\alpha^\ell\,\mathcal{Y}^\alpha\left(g\mathcal{Y}^{k+\ell-\alpha}h\right). \tag{9.24}$$

Next we note that for $s > \frac{n}{p}$, $B_p^s\left(\mathbb{B}_n;\ell^2\right)$ is a Bergman space, hence $M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)} = H^\infty\left(\mathbb{B}_n;\ell^2\right)$. Thus using (6.6) we have for $s > \frac{n}{p}$,

$$g \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \cap H^\infty\left(\mathbb{B}_n;\ell^2\right) = M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \cap M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}.$$

Then, still following the argument in [20], we use the complex interpolation theorem of Beatrous [11] and Ligocka [17] (they prove only the scalar-valued version but the Hilbert space valued version has the same proof),

$$\begin{aligned}\left(B_p^\sigma(\mathbb{B}_n), B_p^{\frac{n}{p}+\varepsilon}(\mathbb{B}_n)\right)_\theta &= B_p^{(1-\theta)\sigma+\theta\left(\frac{n}{p}+\varepsilon\right)}(\mathbb{B}_n), \qquad 0\leq\theta\leq 1,\\ \left(B_p^\sigma\left(\mathbb{B}_n;\ell^2\right), B_p^{\frac{n}{p}+\varepsilon}\left(\mathbb{B}_n;\ell^2\right)\right)_\theta &= B_p^{(1-\theta)\sigma+\theta\left(\frac{n}{p}+\varepsilon\right)}\left(\mathbb{B}_n;\ell^2\right), \qquad 0\leq\theta\leq 1,\end{aligned}$$

to conclude that $g \in M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}$ for all $s \geq \sigma$, and with multiplier norm $\|\mathbb{M}_g\|_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}$ bounded by $\|\mathbb{M}_g\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$. Recall now that

$$\|h\|^p_{B_p^\sigma(\mathbb{B}_n)} = \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\mathcal{Y}^m h(z)\right|^p d\lambda_n(z),$$

and similarly for $\|f\|^p_{B_p^\sigma(\mathbb{B}_n;\ell^2)}$, provided m satisfies

$$\left(\sigma+\frac{m}{2}\right)p > n, \tag{9.25}$$

where $\frac{m}{2}$ appears in the inequality since the derivatives D that can appear in $\mathcal{Y}^m$ only contribute $\left(1-|z|^2\right)^{\frac{1}{2}}$ to the power of $1-|z|^2$ in the integral (see Section 6).

Remark 12. *At this point we recall the convention established in Definitions 6 and 7 that the factors of $1-|z|^2$ that are embedded in the notation for the derivative $\mathcal{Y}^\alpha$ are treated as constants relative to the actual differentiations. In the calculations below, we will adopt* ***the same convention*** *for the factors $\left(1-|z|^2\right)^s$ that we introduce into the integrals. Alternatively, the reader may wish to write out all the derivatives explicitly with the appropriate power of $1-|z|^2$ set aside as is done in* [20].

So we have, keeping in mind Remark 12,

$$\begin{aligned}
&\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma \mathcal{Y}^\alpha\left(g(z)\,\mathcal{Y}^{k+\ell-\alpha}h(z)\right)\right|^p d\lambda_n \\
&\qquad=\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^s \mathcal{Y}^\alpha\left\{g(z)\left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h(z)\right\}\right|^p d\lambda_n \\
&\qquad=\left\|g(z)\left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}.
\end{aligned}$$

Here the function

$$H(z)=\left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h(z)$$

is *not* holomorphic, but we have defined the norm $\|\cdot\|_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}$ on smooth functions anyway. Now we would like to apply a multiplier property of g, and for this we must be acting on a Besov-Sobolev space of *holomorphic* functions, since that is what we get from the complex interpolation of Beatrous and Ligocka. Precisely, we get that $\mathbb{M}_g$ is a bounded operator from $B^s_p(\mathbb{B}_n)$ to $B^s_p\left(\mathbb{B}_n;\ell^2\right)$ for *all* $s\geq\sigma$.

Now we express $\mathcal{Y}^{k+\ell-\alpha}h(z)$ as a sum of terms that are products of a power of $1-|z|^2$ and a derivative $R^iL^jh(z)$ where $i+j=k+\ell-\alpha$ and R is the radial derivative and L denotes a complex tangential derivative $\frac{\partial}{\partial z_j}-\overline{z_j}R$ as in [20]. However, the operators R^iL^j have different weights in the sense that the power of $1-|z|^2$ that is associated with R^iL^j is $\left(1-|z|^2\right)^{i+\frac{j}{2}}$, i.e.

$$\mathcal{Y}^{k+\ell-\alpha}h(z)=\sum\left(1-|z|^2\right)^{i+\frac{j}{2}}R^iL^jh(z).$$

It turns out that to handle the term $\left(1-|z|^2\right)^{i+\frac{j}{2}}R^iL^jh(z)$ we will use that g is a multiplier on $B^s_p(\mathbb{B}_n)$ with

$$s=\sigma+i+\frac{j}{2},$$

an exponent that depends on $i+\frac{j}{2}$ and *not* on $i+j=k+\ell-\alpha$.

Indeed, we have using our "convention" that

$$\begin{aligned}
&\left\| g(z)\left(1-|z|^2\right)^{\sigma-s}\left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)} \\
= &\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^s \mathcal{Y}^\alpha\left\{g(z)\left(1-|z|^2\right)^{\sigma-s}\left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z)\right\}\right|^p d\lambda_n \\
= &\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma+i+\frac{j}{2}} \mathcal{Y}^\alpha\left\{g(z) R^i L^j h(z)\right\}\right|^p d\lambda_n \\
= &\left\| g(z) R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}.
\end{aligned}$$

Now the function $g(z)R^iL^jh(z)$ is holomorphic and $s=\sigma+i+\frac{j}{2}\geq\sigma$ so that we can use that g is a multiplier on $B^s_p(\mathbb{B}_n)=B^s_{p,\alpha}(\mathbb{B}_n)$ (this latter equality holds because $\left(s+\frac{\alpha}{2}\right)p>n$ by (9.25)). The result is that

$$\begin{aligned}
&\left\| g(z) R^i L^j h(z)\right\|^p_{B^s_p(\mathbb{B}_n;\ell^2)} \\
\leq &\left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)}\left\| R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n)} \\
\leq &\left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma+i+\frac{j}{2}}\mathcal{Y}^\alpha R^i L^j h(z)\right|^p d\lambda_n \\
= &\left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma}\mathcal{Y}^\alpha\left[\left(1-|z|^2\right)R\right]^i\left[\sqrt{1-|z|^2}L\right]^j h(z)\right|^p d\lambda_n \\
\leq &\left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma}\mathcal{Y}^{\alpha+i+j} h(z)\right|^p d\lambda_n \\
= &\left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma}\mathcal{Y}^{m} h(z)\right|^p d\lambda_n \\
\leq &\left\|\mathbb{M}_g\right\|^p_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}\left\| h\right\|^p_{B^\sigma_p(\mathbb{B}_n)},
\end{aligned}$$

and the case $M=1$ of Proposition 3 is proved.

Now we turn to the proof of the operator norm inequality (9.21) in Lemma 12. The case $p=2$ is particularly easy:

$$\begin{aligned}
\left\|\mathbb{M}_\varphi f\right\|^2_{B^\sigma_2(\mathbb{B}_n;\otimes^\kappa\ell^2)} &= \int_{\mathbb{B}_n}\left(1-|z|^2\right)^{2\sigma}\sum_{|I|=\kappa-1}\left|\mathcal{Y}^m(\varphi f_I)\right|^2 d\lambda_n \\
&= \sum_{|I|=\kappa-1}\left\|\mathbb{M}_\varphi f_I\right\|^2_{B^\sigma_2(\mathbb{B}_n;\ell^2)} \\
&\leq \left\|\mathbb{M}_\varphi\right\|^2_{B^\sigma_2(\mathbb{B}_n)\to B^\sigma_2(\mathbb{B}_n;\ell^2)}\sum_{|I|=\kappa-1}\left\| f_I\right\|^2_{B^\sigma_2(\mathbb{B}_n)} \\
&= \left\|\mathbb{M}_\varphi\right\|^2_{B^\sigma_2(\mathbb{B}_n)\to B^\sigma_2(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{2\sigma}\sum_{|I|=\kappa-1}\left|\mathcal{Y}^m f_I\right|^2 d\lambda_n \\
&= \left\|\mathbb{M}_\varphi\right\|^2_{B^\sigma_2(\mathbb{B}_n)\to B^\sigma_2(\mathbb{B}_n;\ell^2)}\left\| f\right\|^2_{B^\sigma_2(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)},
\end{aligned}$$

and from this we easily obtain (9.22).

For $p \neq 2$ it suffices to show that

$$\|\mathbb{M}_\varphi\|_{B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\nu)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu\otimes\mathbb{C}^\nu)} \leq C_{n,\sigma,p} \|\mathbb{M}_\varphi\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)} \tag{9.26}$$

for all $\mu,\nu \geq 1$ where the constant $C_{n,\sigma,p}$ is independent of μ,ν. Indeed, both ℓ^2 and $\otimes^{\kappa-1}\ell^2$ are separable Hilbert spaces and so can be appropriately approximated by $\mathbb{C}^\mu$ and $\mathbb{C}^\nu$ respectively. For each $z \in \mathbb{B}_n$ we will view $\varphi(z) \in \mathbb{C}^\mu$ as a column vector and $f(z) \in \mathbb{C}^\nu$ as a row vector so that $(\mathbb{M}_\varphi f)(z)$ is the rank one $\mu \times \nu$ matrix

$$(\mathbb{M}_\varphi f)(z) = \begin{bmatrix} (\varphi_1 f_1)(z) & \cdots & (\varphi_1 f_\nu)(z) \\ \vdots & \ddots & \vdots \\ (\varphi_\mu f_1)(z) & \cdots & (\varphi_\mu f_\nu)(z) \end{bmatrix} = \varphi(z) \odot f(z),$$

where we have inserted the symbol $\odot$ simply to remind the reader that this is *not* the dot product $\varphi(z)\cdot f(z) = f(z)\varphi(z)$ of the vectors $\varphi(z)$ and $f(z)$.

Now we consider a single component X^m of the vector differential operator $\mathcal{Y}^m$ for some $m > 2\left(\frac{n}{p}-\sigma\right)$, which can be chosen independent of μ and ν. The main point in the proof of the lemma is that the matrix $X^m(\mathbb{M}_\varphi f)(z)$ has rank at most $m+1$ independent of μ and ν. Indeed, the Leibniz formula yields

$$X^m(\mathbb{M}_\varphi f)(z) = X^m(\varphi(z)\odot f(z)) = \sum_{\ell=0}^m c_{\ell,m} X^{m-\ell}\varphi(z)\odot X^\ell f(z),$$

where each matrix $X^{m-\ell}\varphi(z)\odot X^\ell f(z)$ is rank one, and where the Hilbert Schmidt norm is multiplicative:

$$\left|X^{m-\ell}\varphi(z)\odot X^\ell f(z)\right| = \left|X^{m-\ell}\varphi(z)\right|\left|X^\ell f(z)\right|.$$

Momentarily fix $0 \leq \ell \leq m$ and define

$$\begin{aligned} T^\ell h(z) &= X^{m-\ell}\varphi(z)\, h(z), \qquad h(z)\in\mathbb{C}, \\ T^\ell g(z) &= X^{m-\ell}\varphi(z)\odot g(z), \qquad g(z)\in\mathbb{C}^\nu. \end{aligned}$$

For $x \in \partial\mathbb{B}_\mu$, which we view as a row vector, define

$$T_x^\ell g(z) = xT^\ell g(z) = x\left(X^{m-\ell}\varphi\right)(z)\odot g(z).$$

Now choose $x(z)\in\partial\mathbb{B}_\mu$ such that $x(z)\left(X^{m-\ell}\varphi\right)(z) = \left|X^{m-\ell}\varphi(z)\right|$ so that

$$T_{x(z)}^\ell g(z) = x(z)\left(X^{m-\ell}\varphi\right)(z)\odot g(z) = \left|X^{m-\ell}\varphi(z)\right| g(z),$$

and hence

$$\left|T_{x(z)}^\ell\left(X^\ell f\right)(z)\right| = \left|X^{m-\ell}\varphi(z)\right|\left|X^\ell f(z)\right| = \left|X^{m-\ell}\varphi(z)\odot X^\ell f(z)\right| = \left|T^\ell\left(X^\ell f\right)(z)\right|.$$

Now we follow the well known argument on page 451 of [26]. For $y \in \partial\mathbb{B}_\nu$, which we view as a column vector, and $g(z)\in\mathbb{C}^\nu$ define the scalars

$$\begin{aligned} g_y(z) &= g(z)\,y, \\ \left(T_{x(z)}^\ell g\right)_y(z) &= T_{x(z)}^\ell g(z)\,y = x(z)\left(X^{m-\ell}\varphi\right)(z)\odot g(z)\,y, \end{aligned}$$

and note that

$$T_{x(z)}^\ell\left(X^\ell f\right)(z)\,y = x(z)\left(X^{m-\ell}\varphi\right)(z)\odot\left(X^\ell f\right)(z)\,y = T_{x(z)}^\ell\left(X^\ell f\right)_y(z).$$

Thus we have with $d\sigma_\nu$ surface measure on $\partial\mathbb{B}_\nu$,

$$\int_{\partial\mathbb{B}_\nu}\left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\,y\right|^p d\sigma_\nu(y)=\left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\right|^p\int_{\partial\mathbb{B}_\nu}\left|\frac{T_{x(z)}^{\ell}\left(X^\ell f\right)(z)}{\left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\right|}\cdot y\right|^p d\sigma_\nu(y),$$

as well as

$$\int_{\partial\mathbb{B}_\nu}\left|\left(X^\ell f\right)_y(z)\right|^p d\sigma_\nu(y)=\left|X^\ell f(z)\right|^p\int_{\partial\mathbb{B}_\nu}\left|\frac{X^\ell f(z)}{\left|X^\ell f(z)\right|}\cdot y\right|^p d\sigma_\nu(y).$$

The crucial observation now is that

$$\int_{\partial\mathbb{B}_\nu}\left|\frac{T_{x(z)}^{\ell}\left(X^\ell f\right)(z)}{\left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\right|}\cdot y\right|^p d\sigma_\nu(y)=\int_{\partial\mathbb{B}_\nu}\left|\frac{X^\ell f(z)}{\left|X^\ell f(z)\right|}\cdot y\right|^p d\sigma_\nu(y)=\gamma_{p,\nu}$$

is *independent* of the row vector in $\partial\mathbb{B}_\nu$ that is dotted with y. Thus we have

$$\begin{aligned}\left|T^\ell\left(X^\ell f\right)(z)\right|^p &= \left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\right|^p=\frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\left|T_{x(z)}^{\ell}\left(X^\ell f\right)(z)\,y\right|^p d\sigma_\nu(y),\\ \left|X^\ell f(z)\right|^p &= \frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\left|\left(X^\ell f\right)_y(z)\right|^p d\sigma_\nu(y).\end{aligned}$$

So with $d\omega_{p\sigma}(z)=\left(1-|z|^2\right)^{p\sigma}d\lambda_n(z)$, we conclude that

$$\begin{aligned}&\int_{\mathbb{B}_n}\left|X^m\left(\mathbb{M}_\varphi f\right)\right|^p d\omega_{p\sigma}(z)\\ \leq\ & C_{n,\sigma,p,m}\sum_{\ell=0}^{m}\int_{\mathbb{B}_n}\left|T^\ell\left(X^\ell f\right)(z)\right|^p d\omega_{p\sigma}(z)\\ =\ & C_{n,\sigma,p,m}\sum_{\ell=0}^{m}\frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\int_{\mathbb{B}_n}\left|x(z)\left(X^{m-\ell}\varphi\right)(z)\left(X^\ell f_y\right)(z)\right|^p d\omega_{p\sigma}(z)\,d\sigma_\nu(y)\\ \leq\ & C_{n,\sigma,p,m}\sum_{\ell=0}^{m}\frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\int_{\mathbb{B}_n}\left|\left(X^{m-\ell}\varphi\right)(z)\left(X^\ell f\right)_y(z)\right|^p d\omega_{p\sigma}(z)\,d\sigma_\nu(y)\\ \leq\ & C_{n,\sigma,p,m}\sum_{\ell=0}^{m}\frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)}^p\int_{\mathbb{B}_n}\left|\left(\mathcal{X}^m f\right)_y(z)\right|^p d\omega_{p\sigma}(z)\,d\sigma_\nu(y)\end{aligned}$$

by the case $M=1$ of Proposition 3, where ℓ^2 there is replaced by $\mathbb{C}^\nu$, g by φ and h by f_y. Now we use the equality

$$\int_{\partial\mathbb{B}_\nu}\left|\left(\mathcal{X}^m f\right)_y(z)\right|^p d\sigma_\nu(y)=\gamma_{p,\nu}\left|\mathcal{X}^m f(z)\right|^p$$

to obtain

$$\begin{aligned}\int_{\mathbb{B}_n}\left|X^m\left(\mathbb{M}_\varphi f\right)\right|^p d\omega_{p\sigma}(z) &\leq C_{n,\sigma,p,m}\left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)}^p\int_{\mathbb{B}_n}\left|\mathcal{X}^m f(z)\right|^p d\omega_{p\sigma}(z)\\ &\leq C_{n,\sigma,p,m}\left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)}^p\left\|f\right\|_{B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)}^p.\end{aligned}$$

Since m depends only on n, σ and p, this completes the proof of (9.26), and hence that of Lemma 12

Finally we return to complete the proof of Proposition 3. We have already proved the case $M=1$. Now we sketch a proof of the case $M=2$ using the multiplier norm inequality (9.21) with $\kappa=2$. By multiplicativity of $|\cdot|$ on tensors, it suffices to show that for $m=\ell_1+\ell_2+k$,

$$\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{\sigma}\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left(\mathcal{Y}^{\ell_2}g\right)\left(\mathcal{Y}^{k}h\right)\right|^p d\lambda_n(z) \leq C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^{2p}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\left\|h\right\|^p_{B_p^\sigma(\mathbb{B}_n)}. \tag{9.27}$$

This time we write using the divergence form of Leibniz' formula on tensor products (c.f. (9.24)),

$$\begin{aligned}\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left(\mathcal{Y}^{\ell_2}g\right)\left(\mathcal{Y}^{k}h\right) &= \left(\mathcal{Y}^{\ell_1}g\right)\otimes\left\{\sum_{\alpha=0}^{\ell_2}c_\alpha^{\ell_2}\mathcal{Y}^{\alpha}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right\}\\ &= \sum_{\alpha=0}^{\ell_2}c_\alpha^{\ell_2}\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left[\mathcal{Y}^{\alpha}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right]\\ &= \sum_{\alpha=0}^{\ell_2}c_\alpha^{\ell_2}\left\{\sum_{\beta=0}^{\ell_1}c_\beta^{\ell_1}\mathcal{Y}^{\beta}\left(g\otimes\mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right)\right\}.\end{aligned}$$

We first use the Hilbert space valued interpolation theorem together with the case $\kappa=2$ of Lemma 12 to show that $g\in M_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)}$ and $g\in M_{B_p^{s_2}(\mathbb{B}_n)\to B_p^{s_2}(\mathbb{B}_n;\ell^2)}$ for appropriate values of s_1 and s_2. Assuming for convenience that $\mathcal{Y}=\left(1-|z|^2\right)R$, and keeping in mind Remark 12, we obtain

$$\begin{aligned}&\left\|g(z)\otimes\left(1-|z|^2\right)^{\sigma-s_1}\mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)}\\ \leq\;&\left\|\mathbb{M}_g\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)}\left\|\left(1-|z|^2\right)^{\sigma-s_1}\mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)}\\ =\;&\left\|\mathbb{M}_g\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{s_1}\mathcal{Y}^{\beta}\left(1-|z|^2\right)^{\sigma-s_1}\mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right|^p d\lambda_n,\end{aligned}$$

which by (9.21) is at most

$$\begin{aligned}&C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{s_2}\mathcal{Y}^{\alpha+\ell_1}\left(g\left(1-|z|^2\right)^{\sigma-s_2}\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right|^p d\lambda_n\\ =\;&C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\left\|g\left(1-|z|^2\right)^{\sigma-s_2}\mathcal{Y}^{k+\ell_2-\alpha}h\right\|^p_{B_p^{s_2}(\mathbb{B}_n;\ell^2)}\\ \leq\;&C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\left\|\mathbb{M}_g\right\|^p_{B_p^{s_2}(\mathbb{B}_n)\to B_p^{s_2}(\mathbb{B}_n;\ell^2)}\left\|\left(1-|z|^2\right)^{\sigma-s_2}\mathcal{Y}^{k+\ell_2-\alpha}h\right\|^p_{B_p^{s_2}(\mathbb{B}_n)}\\ \leq\;&C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^{2p}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}\left\|h\right\|^p_{B_p^\sigma(\mathbb{B}_n)}.\end{aligned}$$

Summing up over α and β gives (9.27).

Repeating this procedure for $M\geq 3$ and using (9.21) with $\kappa=M$ finishes the proof of Proposition 3.

9.4. **Schur's test.** We prove Lemma 10 using Schur's Test as given in Theorem 2.9 on page 51 of [36].

Lemma 13. *Let (X,μ) be a measure space and $H(x,y)$ be a nonnegative kernel. Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Define*

$$\begin{aligned} Tf(x) &= \int_X H(x,y)\, f(y)\, d\mu(y), \\ T^*g(y) &= \int_X H(x,y)\, g(x)\, d\mu(x). \end{aligned}$$

If there is a positive function h on X and a positive constant A such that

$$\begin{aligned} Th^q(x) &= \int_X H(x,y)\, h(y)^q\, d\mu(y) \le Ah(x)^q, \qquad \mu-a.e.x\in X, \\ T^*h^p(y) &= \int_X H(x,y)\, h(x)^p\, d\mu(x) \le Ah(y)^p, \qquad \mu-a.e.y\in X, \end{aligned}$$

then T is bounded on $L^p(\mu)$ with $\|T\|\le A$.

Now we turn to the proof of Lemma 10. The case $c=0$ of Lemma 10 is Lemma 2.10 in [36]. To minimize the clutter of indices, we first consider the proof for the case $c\neq 0$ when $p=2$ and $t=-n-1$. Recall that

$$\begin{aligned} \sqrt{\triangle(w,z)} &= |1-w\overline{z}|\,|\varphi_z(w)|, \\ \psi_\varepsilon(\zeta) &= \left(1-|\zeta|^2\right)^\varepsilon, \end{aligned}$$

and

$$T_{a,b,c}f(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b+n+1}\left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c}}f(w)\,d\lambda_n(w).$$

We will compute conditions on a, b, c and ε such that we have

$$T_{a,b,c}\psi_\varepsilon(z)\le C\psi_\varepsilon(z)\ \text{ and } T^*_{a,b,c}\psi_\varepsilon(w)\le C\psi_\varepsilon(w), \qquad z,w\in\mathbb{B}_n, \tag{9.28}$$

where $T^*_{a,b,c}$ denotes the dual relative to $L^2(\lambda_n)$. For this we take $\varepsilon\in\mathbb{R}$ and compute

$$T_{a,b,c}\psi_\varepsilon(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{n+1+b+\varepsilon}|\varphi_z(w)|^c}{|1-w\overline{z}|^{n+1+a+b}}d\lambda_n(w).$$

Note that the integral is finite if and only if $\varepsilon>-b-1$. Now make the change of variable $w=\varphi_z(\zeta)$ and use that λ_n is invariant to obtain

$$\begin{aligned} T_{a,b,c}\psi_\varepsilon(z) &= \int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{n+1+b+\varepsilon}|\varphi_z(w)|^c}{|1-w\overline{z}|^{n+1+a+b}}d\lambda_n(w) \\ &= \int_{\mathbb{B}_n}F(w)\,d\lambda_n(w)=\int_{\mathbb{B}_n}F(\varphi_z(\zeta))\,d\lambda_n(\zeta) \\ &= \int_{\mathbb{B}_n}\frac{(1-|z|^2)^a\left(1-|\varphi_z(\zeta)|^2\right)^{n+1+b+\varepsilon}|\zeta|^c}{\left|1-\overline{\varphi_z(\zeta)}z\right|^{n+1+a+b}(1-|\zeta|^2)^{n+1}}dV(\zeta). \end{aligned}$$

From the identity (Theorem 2.2.2 in [24]),

$$1-\langle\varphi_a(\beta),\varphi_a(\gamma)\rangle=\frac{(1-\langle a,a\rangle)(1-\langle\beta,\gamma\rangle)}{(1-\langle\beta,a\rangle)(1-\langle a,\gamma\rangle)},$$

we obtain the identities

$$\begin{aligned}1-\varphi_z(\zeta)\overline{z} &= 1-\langle\varphi_z(\zeta),\varphi_z(0)\rangle=\frac{1-|z|^2}{1-\zeta\overline{z}},\\ 1-|\varphi_z(\zeta)|^2 &= 1-\langle\varphi_z(\zeta),\varphi_z(\zeta)\rangle=\frac{\left(1-|z|^2\right)\left(1-|\zeta|^2\right)}{|1-\zeta\overline{z}|^2}.\end{aligned}$$

Plugging these identities into the formula for $T_{a,b,c}\psi_\varepsilon(z)$ we obtain

$$\begin{aligned}(9.29)\ T_{a,b,c}\psi_\varepsilon(z) &= \int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\zeta\overline{z}|^2}\right)^{n+1+b+\varepsilon}|\zeta|^c}{\left|\frac{1-|z|^2}{1-\zeta\overline{z}}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}}dV(\zeta)\\ &= \psi_\varepsilon(z)\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{b+\varepsilon}|\zeta|^c}{|1-\zeta\overline{z}|^{n+1+b-a+2\varepsilon}}dV(\zeta).\end{aligned}$$

Now from Theorem 1.12 in [36] we obtain that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^\alpha}{|1-\zeta\overline{z}|^\beta}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. Provided $c>-2n$ it is now easy to see that we also have

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^\alpha|\zeta|^c}{|1-\zeta\overline{z}|^\beta}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. It now follows from the above that

$$T_{a,b,c}\psi_\varepsilon(z)\le C\psi_\varepsilon(z),\qquad z\in\mathbb{B}_n,$$

if and only if

$$-b-1<\varepsilon<a.$$

Now we turn to the adjoint $T^*_{a,b,c} = T_{b+n+1,a-n-1,c}$ with respect to the space $L^2(\lambda_n)$. With the change of variable $z = \varphi_w(\zeta)$ we have

$$
\begin{aligned}
T^*_{a,b,c}\psi_\varepsilon(w) &= \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\varphi_w(z)|^c}{|1-w\overline{z}|^{n+1+a+b}} d\lambda_n(z) \\
&= \int_{\mathbb{B}_n} G(z)\, d\lambda_n(z) = \int_{\mathbb{B}_n} G(\varphi_w(\zeta))\, d\lambda n(\zeta) \\
&= \int_{\mathbb{B}_n} \frac{\left(1-|\varphi_w(\zeta)|^2\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\zeta|^c}{\left|1-w\overline{\varphi_w(\zeta)}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}} dV(\zeta) \\
&= \int_{\mathbb{B}_n} \frac{\left(\frac{(1-|w|^2)(1-|\zeta|^2)}{|1-\zeta\overline{w}|^2}\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\zeta|^c}{\left|\frac{1-|w|^2}{1-\zeta\overline{w}}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}} dV(\zeta) \\
&= \psi_\varepsilon(w)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{a+\varepsilon-n-1}|\zeta|^c}{|1-\zeta\overline{w}|^{a-b+2\varepsilon-n-1}} dV(\zeta).
\end{aligned} \tag{9.30}
$$

Arguing as above and provided $c > -2n$, we obtain

$$T^*_{a,b,c}\psi_\varepsilon(w) \le C\psi_\varepsilon(w), \qquad w \in \mathbb{B}_n,$$

if and only if

$$-a+n < \varepsilon < b+n+1.$$

Altogether then there is $\varepsilon \in \mathbb{R}$ such that $h = \sqrt{\psi_\varepsilon}$ is a Schur function for $T_{a,b,c}$ on $L^2(\lambda_n)$ in Lemma 13 if and only if

$$\max\{-a+n, -b-1\} < \min\{a, b+n+1\}.$$

This is equivalent to $-2a < -n < 2(b+1)$, which is (7.1) in the case $p = 2, t = -n-1$. Thus Lemma 13 completes the proof that this case of (7.1) implies the boundedness of $T_{a,b,c}$ on $L^2(\lambda_n)$. The converse is easy - see for example the argument for the case $c = 0$ on page 52 of [36].

We now turn to the general case. The adjoint $T^*_{a,b,c}$ relative to the Banach space $L^p(\nu_t)$ is easily computed to be $T^*_{a,b,c} = T_{b-t,a+t,c}$ (see page 52 of [36] for the case $c = 0$). Then from (9.29) and (9.30) we have

$$
\begin{aligned}
T_{a,b,c}\psi_\varepsilon(z) &= \psi_\varepsilon(z)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{b+\varepsilon}|\zeta|^c}{|1-\zeta\overline{z}|^{n+1+b-a+2\varepsilon}} dV(\zeta), \\
T^*_{a,b,c}\psi_\varepsilon(w) &= \psi_\varepsilon(w)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{a+t+\varepsilon}|\zeta|^c}{|1-\zeta\overline{w}|^{a-b+2\varepsilon+t}} dV(\zeta).
\end{aligned}
$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. We apply Schur's Lemma 13 with $h(\zeta) = \left(1-|\zeta|^2\right)^s$ and

$$s \in \left(-\frac{b+1}{q}, \frac{a}{q}\right) \cap \left(-\frac{a+1+t}{p}, \frac{b-t}{p}\right). \tag{9.31}$$

Using Theorem 1.12 in [36] we obtain for h with s as in (9.31) that

$$T_{a,b,c}h^q \leq Ch^q \text{ and } T^*_{a,b,c}h^p \leq Ch^p.$$

Schur's Lemma 13 now shows that $T_{a,b,c}$ is bounded on $L^p(\nu_t)$. Again, the converse follows from the argument for the case $c = 0$ on page 52 of [36].

References

[1] E. Amar, *On the corona problem*, J. of Geom. Anal. **1** (1991), 291-305. 6

[2] C.-G. Ambrozie and D. Timotin, *On an intertwining lifting theorem for certain repoducing kernel Hilbert spaces*, Integr. equ. oper. theory **42** (2002), 373-384. 3

[3] M. Andersson and H. Carlsson, *Wolff-type estimates for $\overline{\partial_b}$ and the H^p corona problem in strictly pseudoconvex domains*, Ark. Mat. **32** (1994), 255-276. 6

[4] M. Andersson and H. Carlsson, *Estimates of the solutions of the H^p and $BMOA$ corona problem*, Math. Ann. **316** (2000), 83-102. 4, 6, 20, 21

[5] M. Andersson and H. Carlsson, *H^p spaces in strictly pseudoconvex domains*, J. Anal. Math. **84** (2001), 335-359. 6

[6] N. Arcozzi, R. Rochberg and E. Sawyer, *Carleson measures and interpolating sequences for Besov spaces on complex balls*, Memoirs A. M. S. **859** (2006), 163 pages. 22, 23, 24, 25, 68

[7] N. Arcozzi, R. Rochberg and E. Sawyer, *Carleson Measures for the Drury-Arveson Hardy space and other Besov-Sobolev Spaces on Complex Balls*, to appear in Advances in Math. 5

[8] W. B. Arveson, *Subalgebras of C^*-algebras III: multivariable operator theory*, Acta Math. **181** (1998), 159-228. 2

[9] J. Agler and J. McCarthy, Pick interpolation and Hilbert function spaces, Graduate Studies in Mathematics **44** (2002), AMS, Providence, RI. 2, 3

[10] J. A. Ball, T. T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, Op. Th. Adv. and App. **122** (2001), 89-138. 3

[11] F. Beatrous, *Estimates for derivatives of holomorphic functions in pseudoconvex domains*, Math. Z. **191** (1986), 91-116. 4, 23, 71, 72

[12] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Annals of Math., **76** (1962), 547-559. 2

[13] P. Charpentier, *Solutions minimales de l'équation $\overline{\partial}u = f$ dans la boule et le polydisque*, Ann. Inst. Fourier (Grenoble) **30** (1980), 121-153. 8, 10, 47

[14] P. P. Fuhrmann, *On the corona theorem and its applications to spectral problems in Hilbert space*, Trans. A.M.S. **132** (1968), 55-66. 2

[15] J. Garnett, *Bounded analytic functions*, Pure and Applied Math. #**96**, Academic Press 1981. 2

[16] S. G. Krantz and S. Y. Li, *Some remarks on the corona problem on strictly pseudoconvex domains*, Illinois J. Math. **39** (1995), 323-349. 6

[17] E. Ligocka, *Estimates of Sobolev norms $\|\cdot\|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Holder spaces of harmonic functions*, Studia Math. **86** (1987), 255-271. 71, 72

[18] K. C. Lin, *The H^p corona theorem for the polydisc*, Trans. Amer. Math. Soc. **341** (1994), 371-375. 6, 19

[19] N. K. Nikol'skiĭ, Operators, functions, and systems: an easy reading, Volume 1: Hardy, Hankel, and Toeplitz. Translated from the French by Andreas Hartmann, Mathematical Surveys and Monographs **92**, A.M.S. Providence, RI, 2002. 2, 4

[20] J. M. Ortega and J. Fabrega, *Pointwise multipliers and decomposition theorems in analytic Besov spaces*, Math. Z. **235** (2000), 53-81. 4, 5, 6, 10, 12, 14, 15, 17, 26, 38, 63, 71, 72, 73

[21] J. M. Ortega and J. Fabrega, *Multipliers in Hardy-Sobolev spaces*, Integr. equ. oper. theory **55** (2006), 535-560. 5

[22] N. Øvrelid, *Integral representation formulas and L^p-estimates for the $\overline{\partial}$-equation*, Math. Scand. **29** (1971), 137-160. 10

[23] M. Rosenblum, *A corona theorem for countably many functions*, Int. Eqs. Op. Theory **3** (1980), 125-137. 2

[24] W. Rudin, Function Theory in the unit ball of C^n, Springer-Verlag 1980. 6, 9, 19, 35, 51, 65, 79

[25] E. Sawyer, Function theory: interpolation and corona problems, Fields Institute monograph series **25**, Amer. Math. Soc. 2009. 4, 19

[26] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N. J., 1993. 75

[27] V. P. Tolokonnikov, *Estimates in Carleson's corona theorem and finitely generated ideals of the algebra H^∞*, Funktsional. Anal. i Prilozhen. **14** (1980), 85-86. 2

[28] V. P. Tolokonnikov, *Estimates in the Carleson corona theorem, ideals of the algebra H^∞, a problem of Szökefalvi-Nagy.* Investigations on linear operators and the theory of functions, XI. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 178-198, 267. 2

[29] V. P. Tolokonnikov, *The corona theorem in algebras of bounded analytic functions*, Amer. Math. Soc. Trans. **149** (1991), 61-93. 5

[30] S. Treil, *Angles between coinvariant subspaces and an operator-valued corona problem. A question of Szökefalvi-Nagy.*, Soviet Math. Dokl. **38** (1989), 394-399. 2

[31] S. Treil and B. D. Wick, *The matrix-valued H^p Corona Problem in the Disk and Polydisk*, J. Funct. Anal. **226** (2005), 138-172. 6

[32] T. Trent, *An H^p corona theorem on the bidisk for infinitely many functions*, Tenth Conference of the International Linear Algebra Society, Linear Algebra Appl. **379** (2004), 213-227. 6

[33] T. T. Trent, *A corona theorem for multipliers on Dirichlet space*, Int. Eqs. Op. Theory **49** (2004), 123-139. 5

[34] T. T. Trent and X. Zhang, *A matricial corona theorem*, Proc. A.M.S. **134** (2006), 2549-2558. 5

[35] N. Th. Varopoulos, *BMO functions and the $\overline{\partial}$ equation*, Pacific J. Math. **71** (1977), 221–273. 2, 6

[36] K. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag 2004.

8, 22, 23, 30, 31, 35, 51, 67, 68, 69, 71, 78, 79, 80, 81

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